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Stability conditions for a Piecewise Deterministic Markov Process

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Abstract

In the present paper we study the stability of a threshold continuos-time model that belongs to the class of Piecewise Deterministic Markov Processes. We derive a sufficient condition on the coefficients of the model to ensure the exponential ergodicity of the process under two different assumptions on the jumps.

Keywords: Threshold process, Compound Poisson Process, Stationary process, Ergodicity.

1 Introduction

We study a Piecewise Deterministic Markov process, as defined in Davis (1984), which can be interpreted as a threshold continuous-time AR(1) model. It generalizes the class of positive storage processes with costant decay allowing the process to assume negative values. In practice we assume jumps to be also negative.

1.1 Definitions

We state here definitions and assumptions that we will use throughout the paper.

- **(D1)** \( \{e_n\}_{n \in \mathbb{N}} \) is a sequence of IID exponential r.v.’s with mean \( \lambda \) and \( e_0 := 0 \).
- **(D2)** \( \{N(t)\}_{t \in \mathbb{R}^+} \) is the Poisson process generated by the inter-arrival times \( \{e_n\} \). We define also the sequence \( \{T_n\}_{n \in \mathbb{N}} \) of the arrival times, that is, \( T_n := e_0 + e_1 + \cdots + e_n \).
- **(D3)** \( \{X_n\}_{n \in \mathbb{N}} \) is a sequence of IID r.v.’s with probability distribution function \( F_X(\cdot) \) and \( X_0 = 0 \). Now we can define the compound Poisson process, related to \( \{N(t)\}, \{P(t)\}_{t \in \mathbb{R}^+} \) as \( P(t) := \sum_{n=1}^{\lfloor N(t) \rfloor} X_n \).
- **(D4)** \( a[y] := a_1 \mathbb{1}_{(0,\infty)}[y] + a_2 \mathbb{1}_{(-\infty,0)}[y] \quad \forall y \in \mathbb{R} \), where \( a_1, a_2 \in \mathbb{R} \) and \( \mathbb{1}_A(\cdot) \) is the usual indicator (or characteristic) function of the set \( A \).
- **(D5)** We indicate with \( \mu(\cdot) \) the Lebesgue measure on \( \mathbb{R} \).

We also assume that

- **(A)** The probability distribution function \( F_X(\cdot) \) is such that

\[
E[X] = 0, \\
E[e^{\alpha X}] < +\infty \quad \alpha > 0.
\]

1.2 The model

We analyze the process \( \{Y(t)\}_{t \in \mathbb{R}^+} \), which trajectories have the following features

- the sequence \( \{X_n\} \) represents the jumps of the process at the times \( \{T_n\} \);
- in between the jumps the process follows an exponential path with rate \( -a[Y(t)] \);
- the value at the time 0 is a realization of a r.v. with an arbitrary probability distribution function \( F_0(\cdot) \), that is \( Y(0) \sim F_0 \).
This model is, therefore, a Piecewise Deterministic Markov Process (PDMP), which is fully described in Davis (1984). Now we build, step by step, the model that generates the values of the process \( \{Y(t)\} \).

- \( t \in (0, T_1) \)
  
  \[ Y(t) = Y(0) e^{-a[Y(0)]t} = Y(0) e^{-a[Y(0)]t} \]

  indeed, multiplying a quantity by the exponential function doesn’t change the sign of the result with respect to the sign of the starting quantity, since \( \exp\{y\} > 0 \quad \forall y \in \mathbb{R} \).

- \( t = T_1 \)
  
  \[ Y(T_1) = Y(0) e^{-a[Y(0)]T_1} + X_1 \]

- \( t \in (T_1, T_2) \)

  \[
  Y(t) = (Y(0) e^{-a[Y(0)]T_1} + X_1) e^{-a[Y(T_1)](t-T_1)} = Y(0) e^{-a[Y(0)]T_1} e^{-a[Y(T_1)](t-T_1)} + X_1 e^{-a[Y(T_1)](t-T_1)} = Y(0) e^{-a[Y(0)]T_1-a[Y(T_1)](t-T_1)} + X_1 e^{-a[Y(T_1)](t-T_1)}
  \]

- \( t = T_2 \)

  \[ Y(T_2) = Y(0) e^{-a[Y(0)]T_1-a[Y(T_1)](T_2-T_1)} + X_1 e^{-a[Y(T_1)](T_2-T_1)} + X_2 \]

- \( t = T_{N(t)} \)

  \[
  Y(T_{N(t)}) = Y(0) e^{-a[Y(0)]T_1-a[Y(T_1)](T_2-T_1)-\cdots-a[Y(T_{N(t)-1})](T_{N(t)}-T_{N(t)-1})} + X_1 e^{-a[Y(T_1)](T_2-T_1)-\cdots-a[Y(T_{N(t)-1})](T_{N(t)}-T_{N(t)-1})} + \cdots + X_{N(t)-1} e^{-a[Y(T_{N(t)-1})](T_{N(t)}-T_{N(t)-1})} + X_{N(t)}
  \]

- \( t \in \mathbb{R}^+ \)

  \[
  Y(t) = Y(0) e^{-a[Y(0)]T_1-a[Y(T_1)](T_2-T_1)-\cdots-a[Y(T_{N(t)})](t-T_{N(t)})} + X_1 e^{-a[Y(T_1)](T_2-T_1)-\cdots-a[Y(T_{N(t)})](t-T_{N(t)})} + \cdots + X_{N(t)} e^{-a[Y(T_{N(t)})](t-T_{N(t)})} = Y(0) \exp \left\{- \sum_{i=1}^{N(t)} a[Y(T_{i-1})](T_i - T_{i-1}) - a[Y(T_{N(t)})](t - T_{N(t)}) \right\} + \sum_{j=1}^{N(t)} X_j \exp \left\{- \sum_{i=j+1}^{N(t)} a[Y(T_{i-1})](T_i - T_{i-1}) - a[Y(T_{N(t)})](t - T_{N(t)}) \right\} = Y(0) e^{-a[Y(0)]T_1-a[Y(T_1)](T_2-T_1)-\cdots-a[Y(T_{N(t)-1})](T_{N(t)}-T_{N(t)-1})} + \]

3
\[ Y(t) = Y(0) e^{-\int_0^t a[Y(s)] \, ds} + \sum_{i=1}^{N(t)} X_i e^{-\int_{T_{i-1}}^{T_i} a[Y(s)] \, ds}. \]  

1.3 Why is it a continuous time threshold AR(1) model?

Since the function \( a[\cdot] \) is a step function (piecewise constant) for \( t \in [0, \infty) \), then the quantity
\[
\sum_{i=1}^{N(t)} a[Y(T_{i-1})] (T_i - T_{i-1}) + a[Y(T_N(t))] (t - T_N(t))
\]
is the sum of the areas of \( N(t) \) rectangles on the \( \mathbb{R}^+ \times \mathbb{R}^+ \) plane, that is, it is the area of the surface between the function \( a[\cdot] \) and the horizontal axis of the \( \mathbb{R}^2 \) plane. But this is exactly the definition of the integral of the function \( a[\cdot] \) in \( [0, \infty) \), so, from (1.2), we can rewrite the process as
\[
Y(t) = Y(0) e^{-\int_0^t a[Y(s)] \, ds} + \sum_{i=1}^{N(t)} X_i e^{-\int_{T_{i-1}}^{T_i} a[Y(s)] \, ds}.
\]

We know that the function
\[
g(t) = g(0) e^{-at} + \int_0^t e^{-a(t-u)} \, dW(u) = g(0) e^{-\int_0^t a \, ds} + \int_0^t e^{-\int_u^t a \, ds} \, dW(u)
\]
is the explicit solution of the stochastic differential equation \( dg(t) = -a g(t) \, dt + dW(t) \), where \( W(t) \) is the standard Brownian motion process.

Since the Compound Poisson process \( P(t) \) is piecewise constant with increments (or decrements) equal to \( X_i \) and the function \( a[\cdot] \) is also piecewise constant, we have
\[
\sum_{i=1}^{N(t)} X_i e^{-\int_{T_{i-1}}^{T_i} a[Y(s)] \, ds} = \int_0^t e^{-\int_u^t a \, ds} \, dP(u)
\]
and, by analogy with the Brownian motion case, (1.4) can be interpreted as the solution of the stochastic differential equation
\[
dY(t) = -a[Y(t)] Y(t) \, dt + dP(t).
\]

In the continuous time model literature, (1.5) has the general form of a threshold AR(1) process.
2 Irreducibility of the process $Y(t)$

In order to state the properties of this process we need to find the form of the generic element of the transition semigroup $(P^t)_{t \in \mathbb{R}^+}$, for $y \in \mathbb{R}$ and $A \in \mathcal{B}(\mathbb{R})$. We remind that $\{e_n\}$ is the sequence of the inter-arrival times of the Poisson process $N(t)$ and $\{X_n\}$ are the jumps of the process $Y(t)$. For an arbitrary $s \in \mathbb{R}^+$ we have

$$P^s(y, A) = P^s_0(y, A) + P^s_1(y, A) + P^s_2(y, A) + \cdots + P^s_m(y, A) + \cdots$$

$$= \mathbb{I}_A[e^{-a|y|s}]\Pr[e_1 > s] +$$

$$+ \int_{-\infty}^{+\infty} (dx_1) \int_0^s (dt_1) \Pr[e_1 = t_1] \Pr[e_2 > s - t_1] \Pr[X_1 = x_1]$$

$$\mathbb{I}_A \left\{ \left( ye^{-a|y|t_1} + x_1 \right) \exp \left\{-a[ye^{-a|y|t_1} + x_1](s - t_1) \right\} \right\} +$$

$$+ \int_{-\infty}^{+\infty} (dx_2) \int_{-\infty}^{+\infty} (dx_1) \int_0^s (dt_1) \int_0^{s-t_1} (dt_2) \Pr[e_3 > s - t_1 - t_2] \Pr[e_2 = t_2] \Pr[e_1 = t_1] \Pr[X_1 = x_1] \Pr[X_2 = x_2]$$

$$\mathbb{I}_A \left\{ \left( ye^{-a|y|t_1} + x_1 \right) e^{-a[e^{-a|y|t_1} + x_1]t_2 + x_2} \right\} \cdot \exp \left\{-a \left( ye^{-a|y|t_1} + x_1 \right) e^{-a[e^{-a|y|t_1} + x_1]t_2 + x_2} (s - t_1 - t_2) \right\} +$$

$$+ \cdots + P^s_m(y, A) + \cdots$$

where

$$P^s_m(y, A) = \Pr[Y(s) \in A| Y(0) = y, N(s) = m]$$

is the transition probability supposed that $m$ jumps happen between times 0 and $s$ and we used the convention that if a r.v. $U$ is absolutely continuous with respect to the Lebesgue measure with density $f_U(\cdot)$, then $\Pr[U = u] = f_U(u)$. By Fubini’s theorem, the above expression of the transition probability, with $A = \mathbb{R}$, gives

$$P^s(y, R) = \Pr[e_1 > s] + \int_{-\infty}^{+\infty} (dx_1) \int_0^s (dt_1) \Pr[e_1 = t_1] \Pr[e_2 > s - t_1] \Pr[X_1 = x_1] +$$

$$+ \int_{-\infty}^{+\infty} (dx_2) \int_{-\infty}^{+\infty} (dx_1) \int_0^s (dt_1) \int_0^{s-t_1} (dt_2) \Pr[e_3 > s - t_1 - t_2] \cdot$$

$$\Pr[e_2 = t_2] \Pr[e_1 = t_1] \Pr[X_1 = x_1] \Pr[X_2 = x_2] + \cdots$$

$$= \sum_{n=0}^{+\infty} \Pr[N(s) = n] = 1.$$  

Now we define the resolvent kernel

$$R(y, A) = \int_0^{+\infty} e^{-s} P^s(y, A) \, ds,$$  

the following function

$$R_1(y, A) = \int_0^{+\infty} e^{-s} P^s_1(y, A) \, ds \leq R(y, A)$$

and, from now on, we assume that
• \((A1)\) The jumps \(\{X_n\}_{n \in \mathbb{N}}\) are a sequence of IID absolutely continuous r.v.’s with lower semicontinuous density function \(f_X(\cdot)\), positive on the whole real line.

A Markov process is called \(\varphi\)-irreducible if for the \(\sigma\)-finite measure \(\varphi\),

\[
\varphi(A) > 0 \Rightarrow \mathbb{E}_y \left[ \int_0^{+\infty} \mathbb{I}_A[Y(t)] \, dt \right] > 0 \quad \forall y \in \mathbb{R}, \quad A \in \mathcal{B}(\mathbb{R}).
\]

We state here also this useful result contained in Proposition 2.2(ii) in Meyn and Tweedie (1993b):

**Proposition 2.1** The process \(Y(t)\) is \(\varphi\)-irreducible \(\iff\) the \(R\)-chain is \(\varphi\)-irreducible.

**Remark 2.1** The \(R\)-chain is the discrete Markov chain that has the resolvent kernel \(R(y, A)\) defined in (2.1) as transition probability function. So, by the irreducibility definition of a discrete Markov chain (see Meyn and Tweedie (1993a)), it is sufficient to check that \(R(y, A) > 0\) whenever \(\varphi(A) > 0\), for \(A \in \mathcal{B}(\mathbb{R})\), to conclude that the continuous time process \(Y(t)\) is \(\varphi\)-irreducible.

We are now able to prove the following result:

**Proposition 2.2** The continuous time Markov process \(\{Y(t)\}_{t \in \mathbb{R}^+}\), defined in (1.4), with assumption \((A1)\), is \(\mu\)-irreducible, where \(\mu(\cdot)\) is the Lebesgue measure on \(\mathbb{R}\).

**PROOF:** We consider an arbitrary set \(A \in \mathcal{B}(\mathbb{R})\) such that \(\mu(A) > 0\). Then, there exists a set \(\hat{A} \in \mathcal{B}(\mathbb{R})\) such that \(\hat{A} \subset A\), with Lebesgue measure \(0 < \mu(\hat{A}) < +\infty\). Furthermore, w.l.o.g., we choose \(\hat{A}\) such that it can be contained in a suitable interval \((c_1, c_2)\) with \(\text{sgn}(c_1) = \text{sgn}(c_2)\) and \(|c_1| < +\infty, \quad |c_2| < +\infty\) (where \(\text{sgn}(\cdot)\) is the sign function). This simply means that the set \(\hat{A}\) is a proper bounded subset of only one of the two sets \(\mathbb{R}^+\) or \(\mathbb{R}^-\).

We recall now the probability of reaching the set \(\hat{A}\) at a certain time \(s\) starting at the time 0 from \(Y(0) = y\), knowing that there was only one jump \(x_1\) at the time \(t_1 \in [0, s]\)

\[
P^s_y(y, \hat{A}) = \int_{-\infty}^{+\infty} (dx_1) \int_0^s (dt_1) \Pr[e_1 = t_1] \Pr[e_2 > s - t_1] \Pr[X_1 = x_1] \cdot \mathbb{I}_\hat{A} \left[ (ye^{-a|y|t_1} + x_1) \exp \left\{ -a|ye^{-a|y|t_1} + x_1|(s - t_1) \right\} \right]. \quad (2.3)
\]

The key to determine the positiveness of \(P^s_y(y, \hat{A})\) is clearly the, inside the integrals, indicator function. We start considering as fixed both the time of the first jump \(t_1\) and the time \(s > t_1\) at which we check if the process \(Y(t)\) is in \(\hat{A}\) with positive probability. Then, by the boundedness of \(\hat{A}\), assumption \((A1)\) and the bijective nature of the exponential function, we know that for each element \(\hat{y} \in \hat{A}\) there exists only one jump \(x_1\) such that

\[
(ye^{-a|y|t_1} + x_1)e^{-a|ye^{-a|y|t_1} + x_1|(s - t_1)} = \hat{y},
\]
that is, the function \( x_1 \mapsto \hat{y} \) is injective. Thus we obtain that, since \( \mu(\hat{A}) > 0 \), there exists a set of jump-values \( \hat{X}_{s,t_1} \) such that \( \mu(\hat{X}_{s,t_1}) > 0 \). The indicator function has value one on that set, hence the outside integral with respect to \( dx_1 \) is on a set of positive Lebesgue measure.

This is also true \( \forall t_1 \leq s \). Indeed, a change in the jump-time causes consequently a change in the jump size, but still the set of the jump-sizes \( \hat{X}_{s,t_1} \) is of positive Lebesgue measure, since we considered an arbitrary fixed time \( t_1 \).

We have therefore that, fixed \( s \), \( P^s_1(y, \hat{A}) > 0 \). The choice of \( s \) was also arbitrary, hence

\[
R_1(y, \hat{A}) = \int_0^{+\infty} e^{-s} P^s_1(y, \hat{A}) \, ds > 0
\]

and, since \( \hat{A} \subset A \), we conclude that

\[
R(y, A) \geq R_1(y, A) > 0
\]

and the resolvent chain is \( \mu \)-irreducible. By Proposition 2.1 the process \( \{Y(t)\} \) is then also \( \mu \)-irreducible and the proof is complete.

\[ \blacksquare \]

**Remark 2.2** In the proof of Proposition 2.2 we showed that

\[
\forall s \in \mathbb{R}^+ \quad P^s_1(y, A) > 0
\]

whenever \( \mu(A) > 0 \), with \( A \in B(\mathbb{R}) \), and therefore, by the definition of \( P^s(y, A) \),

\[
\forall s \in \mathbb{R}^+ \quad P^s(y, A) > 0 .
\]

Then, a straightforward consequence is that every skeleton Markov chain sampled from the process is \( \mu \)-irreducible.

### 3 T-continuity of the process \( Y(t) \)

To prove that the process \( Y(t) \) is T-continuous we need first to prove the following two lemmas.

**Lemma 3.1** Let’s define a sequence \( \{y_n\}_{n \in \mathbb{N}} \) of real numbers. If, for \( y^* \in \mathbb{R} \),

\[
\lim_{n \to +\infty} y_n = y^* < \infty ,
\]

then

\[
\liminf_{n \to +\infty} P^s_1(y_n, A) \geq P^s_1(y^*, A) \quad (3.1)
\]

where \( A \in B(\mathbb{R}) \).
PROOF: In the proof of Proposition 2.2 we showed that for each \( y \in \mathbb{R} \) there exists a set of jumps \( X_{t_1,s} \), with positive Lebesgue measure, and such that for each jump \( x_1 \in X_{t_1,s} \)

\[
(y \ e^{-a|y|t_1} + x_1) e^{-a[y-e^{-a|y|t_1+x_1|(s-t_1)}]} \in \hat{A}
\]

with the set \( \hat{A} \) as in the above mentioned proof and \( s, t_1 \) fixed. The set of jumps \( X_{t_1,s} \) is clearly depending on \( y \) and, from now on, we point this out, by referring to it as \( X_{t_1,s}(y) \) and to the jumps as \( x_1(y) \).

We define here also the following function

\[
G(y) = (y \ e^{-a|y|t_1} + x_1) e^{-a[y-e^{-a|y|t_1+x_1|(s-t_1)}]}
\]

and \( G(y) \in \hat{A} \) because of the suitable choice of every jump \( x_1(y) \).

Let’s consider the case \( y^* \neq 0 \) first.

Then, w.l.o.g., we can assume that \( sgn(y_n) = sgn(y^*) \) \( \forall n \) and therefore \( a[y_n] = a[y^*] = a^* \) \( \forall n \).

Since, by definition, the whole set \( \hat{A} \) is contained in \( \mathbb{R}^+ \) or \( \mathbb{R}^- \), then \( y^* e^{-a^*t_1} + x_1(y^*) \in \mathbb{R}^+ \) or \( \mathbb{R}^- \), respectively, and so does \( y_n e^{-a^*t_1} + x_1(y_n) \) \( \forall n \) as well. Then

\[
a[y_n e^{-a^*t_1} + x_1(y_n)] = a[y^* e^{-a^*t_1} + x_1(y^*)] = a^{**}.
\]

Therefore we are able to restate the function \( G(y) \) defined in (3.2) in the following way:

\[
G(y) = (y d^* + x_1(y)) d^{**}
\]

We notice that the exponential functions involved in (3.3) are not depending on \( y \) and so let’s rewrite it as

\[
G(y) = (y d^* + x_1(y)) d^{**}
\]

with \( d^*, d^{**} \in \mathbb{R}^+ \).

As seen in the proof of Proposition 2.2,

\[
\forall \hat{y} \in \hat{A} \quad \exists! \ x_1(y) \in X_{t_1,s}(y) : G(y) = \hat{y}.
\]

Considering the same specific \( \hat{y} \in \hat{A} \), the function \( x_1(y) \) is invertible and injective and, since \( G(y_n) = G(y^*) = \hat{y} \) for every \( n \), then \( x_1(y_n) \rightarrow x_1(y^*) \) as \( y_n \rightarrow y^* \). Thus we have that, for each fixed jump time \( t_1 \) and \( s \),

\[
\lim_{n \rightarrow +\infty} X_{t_1,s}(y_n) = X_{t_1,s}(y^*)
\]

and, by assumption (A1) and the arbitrary choice of \( t_1 \) and \( s \), we obtain the thesis

\[
\lim inf_{n \rightarrow +\infty} P_t^s(y_n, A) \geq P_t^s(y^*, A).
\]

Now we consider the case \( y^* = 0 \). As in the previous case, since the whole set \( \hat{A} \) is contained in \( \mathbb{R}^+ \) or \( \mathbb{R}^- \), then \( 0 e^{-a|0|t_1} + x_1(0) \in \mathbb{R}^+ \) or \( \mathbb{R}^- \), respectively, and so does \( y_n e^{-a|y_n|t_1} + x_1(y_n) \) \( \forall n \) as well. This implies that

\[
a[y_n e^{-a|y_n|t_1} + x_1(y_n)] = a[x_1(0)] = a^{**}
\]
and we can rewrite $G(y)$ defined in (3.2) as

$$G(y) = (y e^{-a[y]} + x_1(y)) d^* \quad \forall y \neq 0$$

$$G(0) = x_1(0) d^*$$

with $d^* \in \mathbb{R}^+$. No matter what direction the sequence \{\(y_n\)\} converges to 0, to obtain that both $G(y_n) = \hat{y}$ and $G(0) = \hat{y}$, for $\hat{y} \in \hat{A}$, it is necessary that $x_1(y_n) \to x_1(0)$ when $y_n \to 0$. From now on, the proof is the same as in the case $y^* \neq 0$ and the proof is complete.

**Lemma 3.2** Let’s define a sequence \(\{y_n\}_{n \in \mathbb{N}}\) of real numbers. If, for $y^* \in \mathbb{R}$,

$$\lim_{n \to +\infty} y_n = y^* < \infty,$$

then

$$\liminf_{n \to +\infty} R_1(y_n, A) \geq R_1(y^*, A) \quad (3.4)$$

where $A \in \mathcal{B}(\mathbb{R})$ and $R_1(\cdot, \cdot)$ as in (2.2).

**PROOF:** By the Fatou’s lemma we have

$$\liminf_{n \to +\infty} R_1(y_n, A) = \liminf_{n \to +\infty} \int_0^{+\infty} e^{-s} P(s, y_n, A) \, ds$$

$$\geq \int_0^{+\infty} e^{-s} (\liminf_{n \to +\infty} P(s, y_n, A)) \, ds.$$

By Lemma 3.1 we know that

$$\liminf_{n \to +\infty} P(s, y_n, A) \geq P(s, y^*, A)$$

and hence

$$\int_0^{+\infty} e^{-s} (\liminf_{n \to +\infty} P(s, y_n, A)) \, ds \geq \int_0^{+\infty} e^{-s} P(s, y^*, A) \, ds = R_1(y^*, A)$$

and the proof is complete.

We state here the definition of a T-continuous process and then we prove that the process $Y(t)$ is indeed a T-process.

**Definition 3.1** A continuous time Markov process $Y(t)$ is T-continuous (and then it is called T-process) if there exists a sampling distribution $h(\cdot)$ such that the transition kernel

$$K_h(y, A) = \int h(ds) P(s, y, A) \quad (3.5)$$
has a continuous component $T$, with $T(y, \mathbb{R}) > 0$.

$T$ is a continuous component of $K_h$ if $T$ is a substochastic transition kernel satisfying

$$K_h(y, A) \geq T(y, A) \quad y \in \mathbb{R}, \quad A \in \mathcal{B}(\mathbb{R})$$

(3.6)

and $T(\cdot, A)$ is a lower semicontinuous function $\forall A \in \mathcal{B}(\mathbb{R})$.

**Proposition 3.1** The continuous time Markov process \{\(Y(t)\)\}_{t \in \mathbb{R}^+}, defined in (1.4), with assumption (A1), is a $T$-process.

**PROOF:** As transition kernel $K_h$ we choose the resolvent kernel defined in (2.1)

$$R(y, A) = \int_0^{+\infty} e^{-s} P_s(y, A) \, ds,$$

and we will prove that the function defined in (2.2)

$$R_1(y, A) = \int_0^{+\infty} e^{-s} P_1(y, A) \, ds$$

is a continuous component of $R(y, A)$.

By definition, $R_1(y, A)$ is a substochastic transition kernel and also

$$R(y, A) \geq R_1(y, A).$$

By Lemma 3.2 we know that $R_1(\cdot, A)$ is a lower semicontinuous function and in the proof of Proposition 2.2 we showed that $R_1(y, A) > 0$, whenever $\mu(A) > 0$, and, a fortiori, for $A = \mathbb{R}$. Hence Definition 3.1 applies to the process $Y(t)$ and the proof is complete. 

\[ \square \]

## 4 Irreducibility and T-continuity with a different assumption

In this section we consider an alternative assumption on the jump distribution.

- **(A2)** The jumps \{\(X_n\)\} are a sequence of IID absolutely continuous r.v.’s with lower semicontinuous density function, positive on a bounded interval $D_X = (d^-, d^+)$ with $d^- < 0$ and $d^+ > 0$.

**Proposition 4.1** The continuous time Markov process \{\(Y(t)\)\}_{t \in \mathbb{R}^+}, defined in (1.4), with assumption (A2), is $\mu$–irreducible, where $\mu(\cdot)$ is the Lebesgue measure on $\mathbb{R}$. 

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PROOF: First of all we set some notation we will use throughout the proof. Given the initial value \( y \in \mathbb{R} \) we define the sequence \( \{ y^{(n)} \} \) as

\[
\begin{align*}
y^{(0)} &= y \\
y^{(1)} &= y e^{-a|y|t_1} + x_1 \\
y^{(2)} &= y^{(1)} e^{-a|y^{(1)}|t_2} + x_2 \\
&\vdots \\
y^{(n)} &= y^{(n-1)} e^{-a|y^{(n-1)}|t_n} + x_n
\end{align*}
\]

Therefore the probability to reach a set \( A \in \mathcal{B}(\mathbb{R}) \) starting from the initial value \( y \), with \( k \) jumps in between the times 0 and \( s \), is

\[
P^*_k(y, A) = \int_{D_X} (dx_k) \int_{D_X} (dx_{k-1}) \cdots \int_{D_X} (dx_1) \int_0^s (dt_1) \int_0^{s-t_1} (dt_2) \cdots \int_0^{s-\sum_{i=1}^{k-1} t_i} (dt_k) \cdot 
\Pr[e_{k+1} > s - \sum_{i=1}^{k} t_i] \Pr[e_k = t_k] \cdots \Pr[e_1 = t_1] \Pr[X_1 = x_1] \cdots \Pr[X_1 = x_1]
\]

\[I \left[ y^{(k)} \exp \left\{ -a|y^{(k)}|(s - \sum_{i=1}^{k} t_i) \right\} \in A \right]. \quad (4.1)\]

Also, we point out that if \( w \in \mathbb{R} \) then

\[
\forall \, x \in (D_X \cap (\mathbb{R}^+ - \{0\})) \, \exists \, t^* = t(x) > 0 : w e^{-a|w|t} + x \geq w \, \forall \, 0 \leq t \leq t^* \quad (4.2)
\]

and

\[
\forall \, x \in (D_X \cap (\mathbb{R}^- - \{0\})) \, \exists \, t^* = t(x) > 0 : w e^{-a|w|t} + x \leq w \, \forall \, 0 \leq t \leq t^*. \quad (4.3)
\]

We consider now a subset of \( A, \hat{A} \subset A \), defined as in the proof of Proposition 2.2, and choose an element of it, \( \hat{y} \in \hat{A} \). Provided a sufficient number \( k < \infty \) of suitable jumps \( x_i \in D_X \) at the times \( t_i \in [0, t(x_i)] \), the trajectory of the process \( Y(t) \) is such that

\[
y^{(k)} e^{-a|y^{(k)}|(s - \sum_{i=1}^{k} t_i)} = \hat{y}.
\]

Thus we have that the integrals in (4.1) have positive Lebesgue measure supports, since they depend by the inside indicator function, and therefore there exists an integer \( k < \infty \) such that \( P^*_m(y, \hat{A}) > 0 \) for all \( m \geq k \). Indeed, \( k < \infty \) implies that the sequence of jump times \( t(x_i) \) is bounded away from 0 and therefore, by (4.2) and (4.3), the set of the suitable jump sizes has
positive measure. Since $\hat{A} \subset A$ we obtain that $P^s(y, A) > 0$ for an arbitrary, and then for every time $s$, and hence
\[ R(y, A) = \int e^{-s}P^s(y, A) \, ds > 0. \]
Then, by Proposition 2.1, the process $Y(t)$ is $\mu$–irreducible and the proof is complete. 

We state here the definition of \textit{petite} set and a theorem (Meyn and Tweedie (1993b)) we will need in the following to prove the T-continuity of the process.

\textbf{Definition 4.1} A non-empty set $C \in \mathcal{B} (\mathbb{R})$ is $\nu_h$–petite if $\nu_h$ is a non-trivial measure on $\mathcal{B} (\mathbb{R})$, $h(\cdot)$ is a probability measure on $(0, +\infty)$, and $K_h(y, \cdot) \geq \nu_h(\cdot)$ for all $y \in C$. $K_h(\cdot, \cdot)$ is defined as in Definition 3.1.

\textbf{Theorem 4.1} Suppose that
\[ P_y [Y(t) \to \infty] < 1 \]  
for one $y$.
Then:
every compact set is petite $\iff$ the process $Y(t)$ is an irreducible T-process.

\textbf{Proposition 4.2} The continuous time Markov process $\{Y(t)\}_{t \in \mathbb{R}^+}$, defined in (1.4), with assumption (A2), is a T-process.

PROOF: First of all, we define what we are going to use in the proof:

- $y \in \mathbb{R}$ is the initial value of the process $Y(t)$;
- $\{y_n\}_{n \in \mathbb{N}}$ is a sequence of real numbers such that $\lim_{n \to +\infty} y_n = y$;
- $A \in \mathcal{B} (\mathbb{R})$ is a set of positive Lebesgue measure, $\mu(A) > 0$;
- $\hat{A} \subset A$, defined as in the proof of Proposition 2.2;
- $G(y, x, t) = ye^{a|y|t} + x$ with $t > 0$ and $x \in D_X$;
- $\{y^{(n)}\}_{n \in \mathbb{N}}$, defined as in the proof of Proposition 4.1.

In the proof of Proposition 4.1 we showed that
\[ P^s(y, A) > 0 \quad \forall s \in \mathbb{N}, \ y \in \mathbb{R}, \ A \in \mathcal{B} (\mathbb{R}) \]
and that there exists $k < +\infty$ such that $P^s_m(y, A) > 0 \quad \forall m \geq k$.
We analyze first the case where an increasing trajectory is needed to hit the set $\hat{A}$, that is, at
least \( k^+ > 0 \) suitable positive jumps, \( x_i^+ \in X^+ = (D_X \cap \mathbb{R}^+) \), are necessary. We define now the set of the values that a trajectory can reach after the first jump, that is,

\[
Y^{(1)}(y) = \left\{ y^{(1)} \in \mathbb{R} : y^{(1)} = G(y, x_i^+, t_1) \mid x_i \in X^+, \ t_1 \in [0, t(x_i^+)] \right\}, \tag{4.5}
\]

where \( t(x) \) is as in (4.2) and is straightforward to show that \( \mu(Y^{(1)}(y)) > 0 \). In the same way, for each \( y^{(1)} \in Y^{(1)}(y) \), we can define a set \( Y^{(2)}(y^{(1)}) \) and then \( Y^{(3)}(y^{(2)}) \), and so on, obtaining a sequence \( \{Y^{(n)}(y^{(n-1)})\}_{n \in \mathbb{N}} \). By Lemma 3.1 we obtain that

\[
\liminf_{n \to +\infty} P^s_1 \left( y_n, Y^{(1)}(y) \right) \geq P^s_1 \left( y, Y^{(1)}(y) \right) \tag{4.6}
\]

and this result is still valid for every \( m^\text{th} \) step ahead, that is, fixed \( y^{(m)} \in Y^{(m)}(y^{(m-1)}) \) and defined a sequence \( \{y^{(m)}_n\} \) converging to \( y^{(m)} \), then

\[
\liminf_{n \to +\infty} P^s_1 \left( y^{(m)}_n, Y^{(m+1)}(y^{(m)}) \right) \geq P^s_1 \left( y^{(m)}, Y^{(m+1)}(y^{(m)}) \right). \tag{4.7}
\]

Once the trajectory is sufficiently ‘close’ to a subset \( \hat{A}^* \in \hat{A} \), that is, when \( \hat{A}^* \) is reachable in one step, we can apply again the proof of Lemma 3.1 (we are interested obviously only in the subsets of \( \hat{A} \) that have positive Lebesgue measure). Since we put some constraint on the possible trajectories of the process \( Y(t) \) we have, for a suitable \( k < \infty \),

\[
P^s_k \left( y, \hat{A}^* \right) \geq P^s_1 \left( y^{(k+1)}, \hat{A}^* \right) \prod_{i=0}^{k} P^s_1 \left( y^{(i)}, Y^{(i+1)}(y^{(i)}) \right) \tag{4.8}
\]

with \( y^{(0)} = y \). By (4.7) and the boundedness of the probability functions involved we obtain

\[
\begin{align*}
\liminf_{n \to +\infty} P^s_k \left( y_n, \hat{A}^* \right) & \geq \liminf_{n \to +\infty} \left[ P^s_1 \left( y^{(k+1)}_n, \hat{A}^* \right) \prod_{i=0}^{k} P^s_1 \left( y^{(i)}_n, Y^{(i+1)}(y^{(i)}) \right) \right] \\
& \geq \left[ \liminf_{n \to +\infty} P^s_1 \left( y^{(k+1)}_n, \hat{A}^* \right) \right] \left[ \prod_{i=0}^{k} \liminf_{n \to +\infty} P^s_1 \left( y^{(i)}_n, Y^{(i+1)}(y^{(i)}) \right) \right] \\
& \geq P^s_k \left( y^{(k+1)}, \hat{A}^* \right) \prod_{i=0}^{k} P^s_1 \left( y^{(i)}, Y^{(i+1)}(y^{(i)}) \right) > 0. \tag{4.9}
\end{align*}
\]

The case of ‘decreasing’ trajectory is treated in the same way.

Now, if \( C \) is a non-empty compact set and \( A \in \mathcal{B}(\mathbb{R}) \) is such that \( \mu(A) > 0 \), then (4.9) implies that \( \inf_{y \in \mathcal{C}} R(y, A) > 0 \). Indeed, we know from the proof of Proposition 4.1 that the resolvent kernel \( R(y, A) > 0 \ \forall y \in \mathbb{R} \). This means that if \( \inf_{y \in \mathcal{C}} R(y, A) = 0 \) then

\[
\exists y^*, \{y_n\} \in C : R(y^*, A) > 0, \ \lim_{y_n \to y^*} R(y_n, A) = 0.
\]

But this contradicts (4.9) and therefore \( \inf_{y \in \mathcal{C}} R(y, A) > 0 \).

If we define \( \nu_h(\cdot) = \inf_{y \in \mathcal{C}} R(y, A) \), then we have

\[
\nu_h(\cdot) \leq R(y, A) \ \forall y \in \mathcal{C}
\]

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and \( \nu_h(\cdot) \) is a non trivial measure on \( \mathcal{B}(\mathbb{R}) \). Hence the set \( C \) is \( \nu_h - petite \) and so is every other compact set, since the choice of \( C \) was arbitrary.

To apply Theorem 4.1 we need to show that \( \exists y \in \mathbb{R} : P_y[Y(t) \to \infty] < 1 \). First of all let’s explain the meaning of the event \( \{ Y(t) \to \infty \} \) (Meyn and Tweedie (1993b)). Essentially, if
\[
\exists t^* : \forall t > t^*, \forall C \subset \mathbb{R} \quad Y(t) \in C^c,
\]
where \( C \) are compact sets, then we say that the trajectory of the process \( Y(t) \) converges to infinity and we denote this by writing \( \{ Y(t) \to \infty \} \).

Let’s suppose that \( P_y[Y(t) \to \infty] = 1 \ \forall y \in \mathbb{R} \). By the above definition this implies that for every compact set \( C \), a fortiori when \( \mu(C) > 0 \). But in the proof of Proposition 4.1 we showed that
\[
P^s(y, A) > 0 \ \forall s > 0, \quad \forall A \in \mathcal{B}(\mathbb{R}) : \mu(A) > 0.
\]

So we have a contradiction and therefore
\[
\exists y \in \mathbb{R} : P_y[Y(t) \to \infty] < 1.
\]

Now we can apply Theorem 4.1 and conclude that \( Y(t) \) is a T-process.

\[\square\]

## 5 Embedded Markov chains

We define two Markov chains, embedded in the process \( Y(t) \), which will be useful to establish the stability property of the process itself.

We state first a theorem (Meyn and Tweedie (1993a)) which will enable us to show that the above mentioned Markov chains are geometrically ergodic.

**Theorem 5.1** Let \( \{Z_n, n \in \mathbb{N}\} \) be a \( \Psi \)-irreducible and aperiodic chain. If there exist a small set \( K \) and a \( \mathcal{B} \)-measurable function \( V : \mathbb{R} \rightarrow [1, +\infty) \), such that:

1. \( \sup_{z \in K} V(z) < \infty \);
2. \( \sup_{z \in K} E[V(Z_n) \big| Z_{n-1} = z] < \infty \);
3. \( \exists \delta > 0 \) such that \( E[V(Z_n) \big| Z_{n-1} = z] < (1 - \delta) V(z) \ \forall z \notin K \);

then, there exist constants \( r > 1 \) and \( R < +\infty \) such that:
\[
\sum_n r^n \| P^n(z, \cdot) - \pi \|_V \leq R V(z),
\]
where \( \pi \) is the invariant probability and, for any signed measure \( \nu \), we define the \( V \)-norm
\[
\| \nu \|_V = \sup_{g:|g| \leq V} |\nu(g)|.
\]
Furthermore,
\[
\int_{\mathbb{R}} V(t) \, \pi(dt) < \infty.
\]
Also we state here a condition on the coefficients of the process:

- **(C1)** \( a_1 \) and \( a_2 \), defined in (D4), are satisfying
  
  \[
  a_1 > 0, \quad a_2 > 0.
  \]

We start with the discrete time process \( \{Z_n\}_{n \in \mathbb{N}} \), which is the process of the values of the continuous time process \( \{Y(t)\} \) sampled exactly before each jump. So, by (1.1), we have

\[
Z_{N(t)} = Y(T_N(t)) = Y(T_{N(t)}) - X_{N(t)}
\]

with the following recursive formulation

\[
Z_{n+1} = (Z_n + X_n) e^{-a(Z_n + X_n) \epsilon_{n+1}}
\]

and \( Z_0 = Y(0) \).

**Proposition 5.1** The Markov chain \( \{Z_n\} \), defined in (5.2), with assumption (A1) (A2), respectively) is \( \mu \)-irreducible and T-continuous.

**PROOF:** Recalling the definitions (D1) and (D2) in section 1.1, we obtain

\[
\forall m \geq 1 : P^m(\cdot, A) = \int_0^{+\infty} P^s_{m-1}(\cdot + X_0, A) \, Pr[N(s) = m] \, ds
\]

where \( P(\cdot, \cdot) \) is the transition probability of the Markov chain \( \{Z_n\} \).

By Proposition 2.2 and 3.1 (4.1 and 4.2, respectively), the continuous time process \( Y(t) \) is \( \mu \)-irreducible and T-continuous. It is now straightforward to conclude that the Markov chain \( \{Z_n\} \) is also \( \mu \)-irreducible and T-continuous because of the relation (5.3) between the semigroup \( (P^s)_{s \in \mathbb{R}^+} \) and the family \( \{P^n\}_{n \in \mathbb{N}} \).

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Proposition 5.2 The Markov chain \( \{Z_n\} \), defined in (5.2), is geometrically ergodic, provided that condition (C1) is fulfilled.

PROOF: The function \( V(z) = |z| \) satisfies the conditions of Theorem 5.1. Indeed, it is locally bounded and

\[
M(\tilde{z}) = E[|Z_{n+1}| \mid Z_n = \tilde{z}] = E \left[ |\tilde{z} e^{-a|\tilde{z}|X_n} + X_n e^{-a|\tilde{z}|X_n}| \right] \\
\leq |\tilde{z}| E \left[ e^{-a|\tilde{z}|X_n} \right] + E \left[ |X_n| e^{-a|\tilde{z}|X_n} \right] \\
\leq |\tilde{z}| \sup \{ E[e^{-a_1|X_n|}], E[e^{-a_2|X_n|}] \} + \\
+ E [|X_n|] \sup \{ E[e^{-a_1|X_n|}], E[e^{-a_2|X_n|}] \} \\
< \delta_1 |\tilde{z}| + D_1 < \delta_2 |\tilde{z}|, \quad \forall \tilde{z} \in \mathbb{R} : |\tilde{z}| > w \tag{5.4}
\]

where \( D_1 > 0, \delta_1 \in ( \sup \{ E[e^{-a_1|X_n|}], E[e^{-a_2|X_n|}] \}, 1 \) ), \( \delta_1 < \delta_2, w > 0 \), large enough, and

\[
M(\tilde{z}) < \delta_1 |\tilde{z}| + D_1 < +\infty \quad \forall \tilde{z} \in \mathbb{R} : |\tilde{z}| \leq w .
\]

Hence the Markov chain \( \{Z_n\} \) is geometrically ergodic.

We define now the second embedded Markov chain \( \{Z_n\}_{n \in \mathbb{N}} \), which is sampled from the process \( Y(t) \) exactly at the jump times. So we have that \( \{Z_n\} \) is the pre-jump value process, while \( \{Z_n\} \) is the post-jump value process. By (1.1), we have

\[ Z_{N(t)} = Y(T_{N(t)}) \]

with the recursive formulation

\[ Z_{n+1} = Z_n e^{-a[Z_n]X_{n+1}} + X_{n+1} \tag{5.5} \]

and \( Z_0 = Y(0) \).

Proposition 5.3 The Markov chain \( \{Z_n\} \), defined in (5.5), with assumption (A1) (A2), respectively) is \( \mu \)-irreducible and \( T \)-continuous.

PROOF: Reminding the definitions (D1) and (D2) in section 1.1, we obtain

\[
\forall m \geq 1 : \overline{P}^m(\tilde{z}, A) = \int_{0}^{+\infty} P^s_m(\tilde{z}, A) Pr\{N(s) = m\} ds \tag{5.6}
\]

where \( \overline{P}(\cdot, \cdot) \) is the transition probability of the Markov chain \( \{Z_n\} \).

Going ahead as in the proof of Proposition 5.1, we can conclude that \( \{Z_n\} \) is a \( \mu \)-irreducible \( T \)-chain.
Proposition 5.4  The Markov chain \( \{Z_n\} \), defined in (5.5), is geometrically ergodic, provided that condition (C1) is fulfilled.

PROOF: The function \( V(z) = |z| \) satisfies the conditions of Theorem 5.1. Indeed, it is locally bounded and

\[
M(z) = E \left[ |Z_{n+1}| \mid Z_n = z \right] = E \left[ |ze^{-a[Z_{n+1}]} + X_{n+1}| \right] \\
\leq |z| E \left[ e^{-a|Z_{n+1}|} \right] + E \left[ |X_n| \right] \\
\leq |z| \sup \{ E[e^{-a_1e_{n+1}}], E[e^{-a_2e_{n+1}}] \} + E \left[ |X_n| \right] \\
< \delta_1|z| + D_1 < \delta_2|z|, \quad \forall z \in \mathbb{R} : |z| > w \tag{5.7}
\]

where \( D_1 > 0, \delta_i \in (\sup \{ E[e^{-a_1e_{n+1}}], E[e^{-a_2e_{n+1}}] \}, 1) \), \( \delta_1 < \delta_2 \), \( w > 0 \), large enough, and

\[
M(z) < \delta_1|z| + D_1 < +\infty \quad \forall z \in \mathbb{R} : |z| \leq w.
\]

Hence the Markov chain \( \{Z_n\} \) is geometrically ergodic.

6 Stability of the process \( Y(t) \)

In Costa (1990), it is showed that if the Markov chain builded with the post-jump values \( \{Z_n\} \) of a Piecewise Deterministic Markov process \( \{Y(t)\} \) has a unique stationary limit probability distribution, then the continuous time Markov process has a unique stationary limit distribution as well. Our aim in this section is to prove that the process \( \{Y(t)\} \) is also geometrically ergodic. First, we recall, adapted to our case and notation, some main results.

Theorem 6.1 (Theorem 5.5 Davis (1984)) The domain of the extended generator \( \mathcal{A} \) of a Piecewise Deterministic Markov Process consists of the functions \( f(\cdot) \), \( \mathcal{B}(\mathbb{R}) \)-measurable, satisfying

1. \( \forall y \in \mathbb{R} \) the function \( t \rightarrow f(\text{ye}^{-a|y|t}) \) is absolutely continuous for \( t \in \mathbb{R}^+ \).

2. 

\[
f(y) = \int_{\mathbb{R}} f(y + x) \, dF_X(x)
\]

when \( y \to \infty \).

3. 

\[
E \left[ \sum_{T_i \leq t} |f(Y(T_i^+)) - f(Y(T_i^-))| \right] < +\infty
\]

for each \( t \geq 0 \).
For \( f \in \mathcal{D}(\mathcal{A}) \), \( \mathcal{A}f \) is given at \( y \) by

\[
\mathcal{A}f(y) = \frac{d}{dt}f(y) + \lambda \int_{\mathbb{R}} [f(y + x) - f(y)] \, dF_X(x) .
\]  

(6.1)

**Theorem 6.2** (Theorem 3.2 Meyn and Tweedie (1993b)) If all compact subsets of \( \mathbb{R} \) are petite, and there exists a compact set \( K \subset \mathbb{R} \), a constant \( d > 0 \) and a norm-like function \( f(\cdot) \) such that

\[
\mathcal{A}f(y) \leq dI[y \in K], \quad y \in \mathbb{R},
\]  

(6.2)

then \( Y(t) \) is Harris-recurrent.

**Theorem 6.3** (Theorem 6.1 Meyn and Tweedie (1993c)) Suppose that \( Y(t) \) is a right process, and that all compact sets are petite for some skeleton chain. If

\[
\mathcal{A}f(y) \leq -cf(y) + d, \quad \forall y \in \mathbb{R},
\]  

(6.3)

holds with \( f(\cdot) \) norm-like function, \( f(y) \geq 1 \) for each \( y \in \mathbb{R} \), \( c > 0 \) and \( d < +\infty \), then there exists \( \beta < 1 \) and \( B < +\infty \) such that

\[
\|P^s(y, \cdot) - \pi(\cdot)\|_{\mathcal{F}} \leq Bf(y)\beta^s, \quad s \in \mathbb{R}^+, \quad y \in \mathbb{R}.
\]

In the next proposition we define a suitable function \( f(\cdot) \) and we show that it fulfills the conditions stated in Theorem 6.1.

**Proposition 6.1**

\[
f(y) = \begin{cases} 
-2y & \text{if } y < -1 \\
1 + y^2 & \text{if } y \in (-1, 1) \\
2y & \text{if } y > 1
\end{cases}
\]  

(6.4)

is a norm-like, \( \mathcal{B}(\mathbb{R}) \)-measurable function. It satisfies the condition 1.–3. of Theorem 6.1 and is such that \( f(y) \geq 1 \quad \forall y \in \mathbb{R} \).

**PROOF:** It is straightforward to show that the function \( f(\cdot) \) is norm-like and that \( f(y) \geq 1 \quad \forall y \in \mathbb{R} \). Furthermore, \( f \in C^1(\mathbb{R}) \). Indeed,

\[
\lim_{y \downarrow -1} f(y) = \lim_{y \downarrow -1} -2y = 2 = \lim_{y \downarrow -1} 1 + y^2 = \lim_{y \downarrow -1} f(y)
\]

\[
\lim_{y \uparrow 1} f(y) = \lim_{y \uparrow 1} 1 + y^2 = 2 = \lim_{y \uparrow 1} 2y = \lim_{y \uparrow 1} f(y)
\]

\[
\lim_{y \downarrow -1} f'(y) = \lim_{y \downarrow -1} -2 = -2 = \lim_{y \downarrow -1} 2y = \lim_{y \downarrow -1} f'(y)
\]

\[
\lim_{y \uparrow 1} f'(y) = \lim_{y \uparrow 1} 2y = 2 = \lim_{y \uparrow 1} 2 = \lim_{y \uparrow 1} f'(y).
\]
Since \( ye^{-a|y|t} \) is a continuous function with respect to \( t \) and \( f \in C^1(\mathbb{R}) \) then \( f(ye^{-a|y|t}) \) is absolutely continuous for \( t \in \mathbb{R}^+ \).

By assumption (A), we obtain

\[
\lim_{y \to +\infty} y^2 F_X(-y) = 0 \quad \lim_{y \to +\infty} y^2 (1 - F_X(y)) = 0 \\
\lim_{y \to -\infty} y^2 F_X(y) = 0 \quad \lim_{y \to -\infty} y^2 (1 - F_X(-y)) = 0 ,
\]

and every above function is locally bounded.

Now we consider

\[
I(y) = \int_{\mathbb{R}} [f(x + y) - f(y)] \ dF_X(x) = I_1(y) + I_2(y) + I_3(y),
\]

where

\[
I_1(y) = \int_{-\infty}^{-1-y} [-2(x + y) - f(y)] \ dF_X(x) ,
I_2(y) = \int_{-1-y}^{1-y} [1 + (x + y)^2 - f(y)] \ dF_X(x) ,
I_3(y) = \int_{1-y}^{+\infty} [2(x + y) - f(y)] \ dF_X(x) .
\]

For \( y < -1 \)

\[
I_1(y) = \int_{-\infty}^{-1-y} [-2(x + y) - (-2y)] \ dF_X(x) \\
= -2 \int_{-\infty}^{-1-y} x \ dF_X(x) ,
\]

\[
I_2(y) = \int_{-1-y}^{1-y} [1 + (x + y)^2 - (-2y)] \ dF_X(x) \\
= \int_{-1-y}^{1-y} [1 + x^2 + 2xy + y^2 + 2y] \ dF_X(x) \\
= \int_{-1-y}^{1-y} [x^2 + 2xy + (y + 1)^2] \ dF_X(x) \\
= \int_{-1-y}^{1-y} x^2 \ dF_X(x) + 2y \int_{-1-y}^{1-y} x \ dF_X(x) + (y + 1)^2 [F_X(1-y) - F_X(-1-y)] ,
\]

\[
I_3(y) = \int_{1-y}^{+\infty} [2(x + y) - (-2y)] \ dF_X(x) \\
= \int_{1-y}^{+\infty} [2x + 4y] \ dF_X(x) \\
= 2 \int_{1-y}^{+\infty} x \ dF_X(x) + 4y [1 - F_X(1-y)] .
\]

By assumption (A) and (6.5) we have that

\[
|I(y)| < \infty \quad \forall y < -1
\]
and
\[
\lim_{y \to -\infty} |I(y)| < \infty .
\]

For \( y > 1 \)
\[
I_1(y) = \int_{-\infty}^{-1/y} \left[-2(x + y) - 2y\right] dF_X(x) = -2 \int_{-\infty}^{-1/y} x \ dF_X(x) - 4y F_X(-1 - y) ,
\]
\[
I_2(y) = \int_{-1/y}^{1/y} \left[1 + (x + y)^2 - 2y\right] dF_X(x) = \int_{-1/y}^{1/y} \left[x^2 + 2xy + (y - 1)^2\right] dF_X(x)
\]
\[
= \int_{-1/y}^{1/y} x^2 dF_X(x) + 2y \int_{-1/y}^{1/y} x dF_X(x) + (y - 1)^2[F_X(1 - y) - F_X(-1 - y)] ,
\]
\[
I_3(y) = \int_{1/y}^{+\infty} \left[2(x + y) - 2y\right] dF_X(x) = \int_{1/y}^{+\infty} 2x dF_X(x) = \int_{1/y}^{+\infty} x dF_X(x) .
\]

By assumption (A) and (6.5) we have that
\[
|I(y)| < \infty \quad \forall \ y > 1
\]
and
\[
\lim_{y \to +\infty} |I(y)| < \infty .
\]

By assumption (A),
\[
\lim_{y \to -\infty} I_1(y) = \lim_{y \to -\infty} -2 \int_{-\infty}^{-1/y} x \ dF_X(x) = -2E[X] = 0 ,
\]
\[
\lim_{y \to -\infty} I_2(y) = \lim_{y \to -\infty} \left\{ \int_{-1/y}^{1/y} x^2 dF_X(x) + 2y \int_{-1/y}^{1/y} x dF_X(x) + (y - 1)^2[F_X(1 - y) - F_X(-1 - y)] \right\} = 0 ,
\]
\[
\lim_{y \to -\infty} I_3(y) = \lim_{y \to -\infty} \left\{ 2 \int_{1/y}^{+\infty} x dF_X(x) + 4y [1 - F_X(1 - y)] \right\} = 0 ,
\]
\[
\lim_{y \to +\infty} I_1(y) = \lim_{y \to +\infty} -2 \int_{-\infty}^{-1/y} x \ dF_X(x) - 4y F_X(-1 - y) = 0 ,
\]
\[
\lim_{y \to +\infty} J_2(y) = \lim_{y \to +\infty} \left\{ \int_{-\infty}^{1-y} x^2 \, dF_X(x) + 2y \int_{-1-y}^{1-y} x \, dF_X(x) + (y-1)^2 |F_X(1-y) - F_X(-1-y)| \right\} = 0,
\]
and therefore we obtain that condition 2. in Theorem 6.1 is fulfilled.

Now we consider
\[
J(y) = \int_{E_1} |f(x+y) - f(y)| \, dF_X(x) = J_1(y) + J_2(y) + J_3(y),
\]
where
\[
J_1(y) = \int_{-\infty}^{-1-y} \left| -2(x+y) - f(y) \right| \, dF_X(x),
\]
\[
J_2(y) = \int_{1-y}^{1+y} \left| 1 + (x+y)^2 - f(y) \right| \, dF_X(x),
\]
\[
J_3(y) = \int_{1-y}^{+\infty} \left| 2(x+y) - f(y) \right| \, dF_X(x).
\]

For \( y < -1 \)
\[
J_1(y) = \int_{-\infty}^{-1-y} \left| -2(x+y) - (-2y) \right| \, dF_X(x)
= 2 \int_{-\infty}^{-1-y} |x| \, dF_X(x)
\leq 2E[|X|],
\]
\[
J_2(y) = \int_{-1-y}^{1-y} \left| 1 + (x+y)^2 - (-2y) \right| \, dF_X(x)
= \int_{-1-y}^{1-y} \left| 1 + x^2 + 2xy + y^2 + 2y \right| \, dF_X(x)
= \int_{-1-y}^{1-y} \left| x^2 + 2xy + (y+1)^2 \right| \, dF_X(x)
\leq \int_{-1-y}^{1-y} x^2 \, dF_X(x) + 2|y| \int_{-1-y}^{1-y} |x| \, dF_X(x) + (y+1)^2 \left[ F_X(1-y) - F_X(-1-y) \right]
\leq E\left[ X^2 \right] + 2|y| \int_{-1-y}^{1-y} |x| \, dF_X(x) + (y+1)^2 \left[ F_X(1-y) - F_X(-1-y) \right],
\]
\[
J_3(y) = \int_{1-y}^{+\infty} \left| 2(x+y) - (-2y) \right| \, dF_X(x)
= \int_{1-y}^{+\infty} \left| 2x + 4y \right| \, dF_X(x)
\leq 2 \int_{1-y}^{+\infty} |x| \, dF_X(x) + 4|y| \left| 1 - F_X(1-y) \right|.
\]

By assumption (A) and (6.5) we have that
\[
J(y) < \infty \quad \forall \ y < -1
\]
and

\[ \lim_{y \to -\infty} J(y) < \infty. \]

Therefore there exists \( J^{(1)} < +\infty \) such that

\[ J(y) < J^{(1)} \quad \forall y < -1. \]

For \( y \in (-1, 1) \)

\[ J_1(y) = \int_{-\infty}^{-1-y} \left| -2(x + y) - (1 + y^2) \right| dF_X(x) \]
\[ = \int_{-\infty}^{-1-y} \left| -2x - 2y - 1 - y^2 \right| dF_X(x) \]
\[ \leq 2 \int_{-\infty}^{-1-y} |x| \, dF_X(x) + (1 + y)^2 F_X(-1 - y) \]
\[ \leq 2E\|X\| + (1 + y)^2 \]
\[ \leq 2E\|X\| + 4, \]

\[ J_2(y) = \int_{1-y}^{1} \left| 1 + (x+y)^2 - (1+y^2) \right| dF_X(x) \]
\[ = \int_{1-y}^{1} \left| 1 + x^2 + 2xy + y^2 - 1 - y^2 \right| dF_X(x) \]
\[ = \int_{1-y}^{1} \left| x^2 + 2xy \right| dF_X(x) \]
\[ \leq \int_{1-y}^{1} x^2 \, dF_X(x) + 2|y| \int_{1-y}^{1} |x| \, dF_X(x) \]
\[ \leq E\left[X^2\right] + 2|y|E\|X\| \]
\[ \leq E\left[X^2\right] + 2E\|X\|, \]

\[ J_3(y) = \int_{1-y}^{+\infty} \left| 2(x + y) - (1 + y^2) \right| dF_X(x) \]
\[ = \int_{1-y}^{+\infty} \left| 2x + 2y - 1 - y^2 \right| dF_X(x) \]
\[ \leq 2 \int_{1-y}^{+\infty} |x| \, dF_X(x) + (y - 1)^2 [1 - F_X(1 - y)] \]
\[ \leq 2E\|X\| + (y - 1)^2 \]
\[ \leq 2E\|X\| + 4. \]

By assumption (A) we have

\[ J(y) < \infty \quad \forall y \in (-1, 1) \]

and there exists

\[ J^{(2)} = 2E\|X\| + \max\left\{4, E\left[X^2\right]\right\} < +\infty \]
such that
\[ J(y) < J^{(2)} \quad \forall \ y \in (-1, 1). \]

For \( y > 1 \)
\[
J_1(y) = \int_{-\infty}^{-1+y} |-2(x+y) - 2y| \ dF_X(x)
\leq 2 \int_{-\infty}^{-1+y} |x| \ dF_X(x) + 4y F_X(-1-y)
\leq 2 ||X|| + 4y F_X(-1-y),
\]
\[
J_2(y) = \int_{-1-y}^{1-y} |1 + (x+y)^2 - 2y| \ dF_X(x)
= \int_{-1-y}^{1-y} |1 + x^2 + 2xy + y^2 - 2y| \ dF_X(x)
= \int_{-1-y}^{1-y} |x^2 + 2xy + (y-1)^2| \ dF_X(x)
\leq \int_{-1-y}^{1-y} x^2 \ dF_X(x) + 2y \int_{-1-y}^{1-y} |x| \ dF_X(x) + (y-1)^2 [F_X(1-y) - F_X(-1-y)]
\leq E[X^2] + 2y \int_{-1-y}^{1-y} |x| \ dF_X(x) + (y-1)^2 [F_X(1-y) - F_X(-1-y)],
\]
\[
J_3(y) = \int_{1-y}^{+\infty} |2(x+y) - 2y| \ dF_X(x)
= \int_{1-y}^{+\infty} 2|x| \ dF_X(x)
\leq 2E[|X|].
\]

By assumption (A) and (6.5) we have that
\[ J(y) < \infty \quad \forall \ y > 1 \]
and
\[ \lim_{y \to +\infty} J(y) < \infty. \]

Therefore there exists \( J^{(3)} < +\infty \) such that
\[ J(y) < J^{(3)} \quad \forall \ y > 1. \]

If we define
\[ J = \max \{J^{(1)}, J^{(2)}, J^{(3)}\} \]
we obtain that \( J(y) < J < +\infty \ \forall \ y \in \mathbb{R} \) and condition 3. is satisfied, since the sum involved in the condition has a finite number of terms. The proof is now complete. \( \blacksquare \)

Recalling (6.1), we can build now the extended generator of the process \( Y(t) \).
Proposition 6.2  The extended generator of the process \( Y(t) \) is

\[
\mathcal{A}f(y) = q_1(y) + q_2(y)
\]

where

\[
q_1(y) = \begin{cases} 
-a[y]f(y) & y < -1 \\
-2a[y]f(y) + 2a[y] & y \in (-1, 1) \\
-a[y]f(y) & y > 1 
\end{cases}
\]

and

\[
|q_2(y)| \leq \Delta < +\infty
\]

PROOF: Recalling the proof of the Proposition 6.1, we obtained the value of

\[
I(y) = \int_{\mathbb{R}} [f(y + x) - f(y)] \, dF_X(x)
\]

for every \( y \in \mathbb{R} \). We showed that, for \( y \notin (-1, 1) \),

\[
|I(y)| < +\infty.
\]

We analyze now the remaining case, \( y \in (-1, 1) \).

\[
I_1(y) = \int_{-\infty}^{-1-y} \left[-2(x+y) - (1+y^2)\right] \, dF_X(x)
\]

\[
= \int_{-\infty}^{-1-y} \left[-2x - 2y - 1 - y^2\right] \, dF_X(x)
\]

\[
= -2 \int_{-\infty}^{-1-y} x \, dF_X(x) - (1+y)^2 F_X(-1-y)
\]

\[
I_2(y) = \int_{-1-y}^{1-y} \left[1 + (x+y)^2 - (1+y^2)\right] \, dF_X(x)
\]

\[
= \int_{-1-y}^{1-y} \left[1 + x^2 + 2xy + y^2 - 1 - y^2\right] \, dF_X(x)
\]

\[
= \int_{-1-y}^{1-y} \left[x^2 + 2xy\right] \, dF_X(x)
\]

\[
= \int_{-1-y}^{1-y} x^2 \, dF_X(x) + 2y \int_{-1-y}^{1-y} x \, dF_X(x)
\]

\[
I_3(y) = \int_{1-y}^{+\infty} \left[2(x+y) - (1+y^2)\right] \, dF_X(x)
\]

\[
= \int_{1-y}^{+\infty} \left[2x + 2y - 1 - y^2\right] \, dF_X(x)
\]

\[
= 2 \int_{1-y}^{+\infty} x \, dF_X(x) - (y-1)^2 [1 - F_X(1-y)]
\]

By assumption (A) and since \( y \) belongs to a bounded set, we have

\[
|I(y)| < \infty \quad \forall y \in (-1, 1)
\]

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and
\[ \sup_{y \in (-1,1)} |I(y)| = |I| < +\infty. \]

Summarizing, we have
\[ \left| \int_{\mathcal{F}_1} [f(x + y) - f(y)] \, dF_x(x) \right| < \infty \]
and therefore, recalling the expression of the extended generator (6.1) in Theorem 6.1, we obtain
\[ |\lambda I(y)| = |q_2(y)| < +\infty. \]

The derivative with respect to \( t \) of the function \( g(w, t) = we^{-a[w]t} \) is
\[ \frac{dg}{dt}(w, t) = -a[w]we^{-a[w]t} = -a[w]g(w, t). \tag{6.8} \]

By (1.3), we have that the trajectory of the process is
\[ Y(t) = g \left( Y(T_{N(t)}), t - T_{N(t)} \right) \quad \forall \ t \in [T_{N(t)}, T_{N(t)+1}) \tag{6.9} \]
and the steepness of the deterministic path in between the jumps is therefore the same as the one of the function \( g(\cdot, \cdot) \) calculated in \( Y(T_{N(t)}) \) and time starting from \( t - T_{N(t)} \). Hence,
\[ \frac{dY(t)}{dt} = -a \left[ Y(T_{N(t)}) \right] Y(T_{N(t)})e^{-a[Y(T_{N(t)})](t-T_{N(t)})} = -a[Y(T_{N(t)})]Y(t) \]
and, since \( sgn(Y(T_{N(t)})) = sgn(Y(t)) \),
\[ \frac{dY(t)}{dt} = -a[Y(t)]Y(t). \tag{6.10} \]

With very simple calculus we obtain
\[ \frac{df(Y(t))}{dt} = \frac{df(y)}{dy} \frac{dY(t)}{dt} \]
and therefore
\begin{itemize}
  \item \( Y(t) < -1 \)
    \[ f(Y(t)) = -2Y(t) \]
    \[ \frac{df(Y(t))}{dt} = -2(-a[Y(t)])Y(t) = -a[Y(t)]f(Y(t)) \]
  \item \( Y(t) \in (-1,1) \)
    \[ f(Y(t)) = 1 + Y(t)^2 \]
    \[ \frac{df(Y(t))}{dt} = 2Y(t)(-a[Y(t)])Y(t) = -2a[Y(t)]f(Y(t)) + 2a[Y(t)] \]
  \item \( Y(t) > 1 \)
    \[ f(Y(t)) = 2Y(t) \]
    \[ \frac{df(Y(t))}{dt} = 2(-a[Y(t)])Y(t) = -a[Y(t)]f(Y(t)). \]
\end{itemize}
The proof is now complete.

**Proposition 6.3** If assumptions (A), (A1) ((A2), resp.) and condition (C1) are satisfied then the extended generator, defined in Proposition 6.2, is such that

\[ Af(y) \leq -cf(y) + D , \quad \forall y \in \mathbb{R} , \tag{6.11} \]

where \( c > 0 \), \( 0 < D < +\infty \).

**PROOF:** The condition (C1) ensures that \(-a[y] < 0\) for each \( y \in \mathbb{R} \) and, recalling (6.6) and (6.7) in Proposition 6.2, the proof is complete.

We are now able to state the stability results for the Markov process \( Y(t) \).

**Proposition 6.4** The Piecewise Deterministic Markov process \( Y(t) \), defined in (1.4), is Harris-recurrent if assumptions (A), (A1) ((A2), resp.) and condition (C1) are satisfied.

**PROOF:** Since the function \( f(\cdot) \) is locally bounded, positive, symmetric and strictly increasing on \( \mathbb{R}^+ \), by Proposition 6.3, there exists \( r > 0 \) such that

\[-cf(y) + D < 0 \quad \forall y \notin K = [-r,r] \]

and hence

\[ Af(y) \leq D \mathbb{1}_K(y) \quad \forall y \in \mathbb{R} . \]

The process \( Y(t) \) is T-continuous by Proposition 3.1 (4.2, resp.) and therefore all compact subsets of \( \mathbb{R} \) are petite. We can therefore apply Theorem 6.2 and the proof is complete.

**Proposition 6.5** The Piecewise Deterministic Markov process \( Y(t) \), defined in (1.4), is positive Harris-recurrent if assumptions (A), (A1) ((A2), resp.) and condition (C1) are satisfied.

**PROOF:** We already pointed out, at the beginning of this section, that the process \( Y(t) \) has a stationary limit probability distribution. A Harris-recurrent process with finite stationary measure is, by definition, a positive Harris-recurrent process.

**Proposition 6.6** The Piecewise Deterministic Markov process \( Y(t) \), defined in (1.4), is ergodic if assumptions (A), (A1) ((A2), resp.) and condition (C1) are satisfied.
PROOF: By Proposition 6.5, the process $Y(t)$ is positive Harris-recurrent. In the proof of Proposition 2.2 (4.1, resp.) we showed that, for $y \in \mathbb{R}$ and $A \in \mathcal{B}(\mathbb{R})$,

$$P^s(y, A) > 0 \quad \forall s \in \mathbb{R}^+$$

whenever $\mu(A) > 0$. This implies that every sampled skeleton chain is $\mu$–irreducible as well. Hence, we can apply Theorem 6.1 in Meyn and Tweedie (1993b) and conclude that the process is ergodic.

\begin{proposition}
The Piecewise Deterministic Markov process $Y(t)$, defined in (1.4), is exponentially ergodic if assumptions (A), (A1) ((A2), resp.) and condition (C1) are satisfied.
\end{proposition}

PROOF: Since $Y(t)$ is ergodic (Prop. 6.6) and T-continuous, then, by Proposition 6.1 in Meyn and Tweedie (1993b) we have that all compac subsets of $\mathbb{R}$ are petite for every skeleton chain sampled from the process. Recalling (6.11) in Proposition 6.3, we can apply now Theorem 6.3 and the proof is complete.

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\section*{References}


