Ivan Ginchev, Matteo Rocca

On constrained set-valued optimization

2007/10
In questi quaderni vengono pubblicati i lavori dei docenti della Facoltà di Economia dell’Università dell’Insubria. La pubblicazione di contributi di altri studiosi, che abbiano un rapporto didattico o scientifico stabile con la Facoltà, può essere proposta da un professore della Facoltà, dopo che il contributo sia stato discusso pubblicamente. Il nome del proponente è riportato in nota all'articolo. I punti di vista espressi nei quaderni della Facoltà di Economia riflettono unicamente le opinioni degli autori, e non rispecchiano necessariamente quelli della Facoltà di Economia dell'Università dell'Insubria.

These Working papers collect the work of the Faculty of Economics of the University of Insubria. The publication of work by other Authors can be proposed by a member of the Faculty, provided that the paper has been presented in public. The name of the proposer is reported in a footnote. The views expressed in the Working papers reflect the opinions of the Authors only, and not necessarily the ones of the Economics Faculty of the University of Insubria.
On constrained set-valued optimization

Ivan Ginchev† Matteo Rocca‡

Abstract

The set-valued optimization problem \( \min_{C} F(x), \ G(x) \cap (-K) \neq \emptyset \), is considered, where \( F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m \) and \( G : \mathbb{R}^n \rightrightarrows \mathbb{R}^p \) are set-valued functions, and \( C \subset \mathbb{R}^m \) and \( K \subset \mathbb{R}^p \) are closed convex cones. Two type of solutions, called \( w \)-minimizers (weakly efficient points) and \( i \)-minimizers (isolated minimizers), are treated. In terms of the Dini set-valued directional derivative first-order necessary conditions for a point to be a \( w \)-minimizer, and first-order sufficient conditions for a point to be an \( i \)-minimizer are established, both in primal and dual form.

Key words: Set-valued optimization, First-order optimality conditions, Dini derivatives.

Math. Subject Classification: 49J53, 49J52, 90C29, 90C30, 90C46.

1 Introduction

The constrained set-valued optimization problem (svp)

\[
\min_{C} F(x), \quad G(x) \cap (-K) \neq \emptyset,
\]

is considered, where \( F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m \) and \( G : \mathbb{R}^n \rightrightarrows \mathbb{R}^p \) are set-valued functions (svf) with non-empty values, and \( C \subset \mathbb{R}^m \) and \( K \subset \mathbb{R}^p \) are closed convex cones. First order optimality conditions in terms of the Dini set-valued directional derivative are derived. The obtained results generalize those of [3] from vector to set-valued problem, and of [1] from unconstrained to constrained problem. Recently optimality conditions for svp are studied mainly by means of epiderivatives, e. g. in [4], [5] and [2]. We consider the optimality conditions based on directional derivatives as certain alternative of those based on epiderivatives. Some comparison of the two methods is done in [1].

†University of Insubria, Department of Economics, Via Monte Generoso 71, 21100 Varese, Italy, iginchev@eco.uninsubria.it.
‡University of Insubria, Department of Economics, Via Monte Generoso 71, 21100 Varese, Italy, mrocca@eco.uninsubria.it.
2 Preliminaries

The space $\mathbb{R}^k$ is considered with the usual topology. It is generated by an arbitrary norm. Since all the norms in $\mathbb{R}^k$ are equivalent, we make use of this sometimes choosing a special norm. The dual pairing in $\mathbb{R}^k$ is a denoted $\langle \cdot , \cdot \rangle$. It is a mapping $\mathbb{R}^k \times (\mathbb{R}^k)^* \to \mathbb{R}$, where $(\mathbb{R}^k)^*$ stands for the dual of $\mathbb{R}^k$. In fact $(\mathbb{R}^k)^*$ can be identified with $\mathbb{R}^k$ as linear and topological spaces, but when $\mathbb{R}^k$ is considered with a norm, the dual space $(\mathbb{R}^k)^*$ is supplied with the dual norm. When $\mathbb{R}^k$ is supplied with an Euclidean norm, then $(\mathbb{R}^k)^*$ can be identified with $\mathbb{R}^k$ also as a norm space. The notations $B_k$ and $\bar{B}_k$ are used for the open and closed unit balls, and $B_k(x^0)$ and $\bar{B}_k(x^0)$ for the open and closed unit balls with center $x^0$.

For a given closed convex cone $C \subset \mathbb{R}^k$ its positive polar cone is defined by $M = \{ \xi \in (\mathbb{R}^k)^* \mid \langle \xi, x \rangle \geq 0 \text{ for all } y \in M \}$. When $x^0 \in M$ we put $M'[x^0] = \{ \xi \in M' \mid \langle \xi, x^0 \rangle = 0 \}$ and $M[x^0] = (M'[x^0])'$. It holds $M \subset M[x^0]$.

When $\mathbb{R}^k$ is considered with a concrete norm, the distance from a point $x \in \mathbb{R}^k$ to a set $A \subset \mathbb{R}^k$ is given by $d(x, A) = \inf \{ \| x - y \| \mid a \in A \}$. The oriented distance from $x$ to $A$ is defined by $D(x, A) = d(x, A) - d(x, \mathbb{R}^k \setminus A)$. When $C \subset \mathbb{R}^k$ is a proper closed convex cone, then $D(x, M) = \sup \{ \| \xi \| \mid \xi \in M', \| \xi \| = 1 \}$. We define the oriented distance $D(P, A)$ from a set $P \subset \mathbb{R}^k$ to the set $A \subset \mathbb{R}^k$ putting $D(P, A) = \inf \{ D(x, A) \mid x \in P \}$.

Using the oriented distance we introduce the following notion. Let $C \subset \mathbb{R}^k$ be a cone and let $a$ be a real number. Then we put $M(a) = \{ x \in \mathbb{R}^k \mid D(x, M) \leq a \| x \| \}$. The weakly efficient frontier (w-frontier) $w$-$\text{Min}_C A$ and the properly efficient frontier (p-frontier) $p$-$\text{Min}_C A$ of $A$ are defined respectively by $w$-$\text{Min}_C A = \{ x \in A \mid A \cap (x - \text{int} M) = \emptyset \}$ and $p$-$\text{Min}_C A = \{ x \in A \mid \exists a \in (0, 1) : A \cap (x - M(a)) = \{ x \} \}$.

The set of the feasible points of svp (1) is defined by $G = \{ x \in \mathbb{R}^n \mid G(x) \cap (-K) \neq \emptyset \}$. Further $\mathcal{N}(x^0)$ denotes the family of the neighbourhoods of $x^0$. We deal with local solutions of (1), which in any case are pairs $(x^0, y^0)$, $y^0 \in F(x^0)$, with $x^0$ feasible. Here we use the following concepts of solutions for problem (1). The pair $(x^0, y^0)$, $x^0 \in \mathbb{R}^n$, $y^0 \in F(x^0)$, is said a $w$-minimizer (weakly efficient point) if there exists $U \in \mathcal{N}(x^0)$ such that $x \in U \cap G$ implies $F(x) \cap (y^0 - \text{int} C) = \emptyset$ (then necessary $y^0 \in w$-$\text{Min}_C F(x^0)$).

The pair $(x^0, y^0)$ is said an $i$-minimizer (isolated minimizer) if there exists $U \in \mathcal{N}(x^0)$ and a constant $A > 0$ such that $D(F(x) - y^0, -C) \geq A \| x - x^0 \|$ and $y^0 \in p$-$\text{Min}_C F(x^0)$ for $x \in U \cap G$ (the concept of $i$-minimizer is norm-independent, since all norms in finite-dimensional spaces are equivalent).

The svf $\Phi : \mathbb{R}^n \to \mathbb{R}^k$ is said locally Lipschitz at $x^0 \in \mathbb{R}^n$, if there exists $U \in \mathcal{N}(x^0)$ and a constant $L > 0$, such that for $x^1, x^2 \in U$ it holds $\Phi(x^2) - \Phi(x^1) + L \| x^2 - x^1 \| \in \bar{B}_k$. The svf $\Phi$ is said locally Lipschitz, if it is locally Lipschitz at each $x^0 \in \mathbb{R}^n$. Given a cone $M \subset \mathbb{R}^k$, we say that $\Phi$ is locally $M$-Lipschitz at $x^0$ if the svf $x \sim \Phi(x) + M$ is locally Lipschitz at $x^0$. The svf $\Phi$ is said locally $M$-Lipschitz, if it is locally $M$-Lipschitz at each point $x^0$.

The cone $M \subset \mathbb{R}^k$ is said pointed if $M \cap (-M) = \{ 0 \}$, and $M$ is contained in a half-space of $\mathbb{R}^k$. If $\mathbb{R}^k$ is supplied with an Euclidean norm, then the cone $M$ is said non-obtuse, if $\langle x^1, x^2 \rangle \geq 0$ for all $x^1, x^2 \in M$. Each non-obtuse cone is pointed. The following result converts in some sense this statement.

**Lemma 1** ([1]) If $M \subset \mathbb{R}^k$ is a pointed closed convex cone, then there exists an Euclidean norm in $\mathbb{R}^k$ with respect to which $M$ is a non-obtuse cone.
The next lemma is essential for the proof of the optimality conditions for (1).

**Lemma 2 ([1])** Let $M \subset \mathbb{R}^k$ be a non-obtuse closed convex cone and $M \setminus \{0\} \neq \emptyset$. Let the svf $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^k$ be $C$-Lipschitz with constant $L$ in $U \in \mathcal{N}(x^0)$ and $y^0 \in \Phi(x^0)$. Suppose that for some $\sigma \in (0, 1/2)$ it holds $\Phi(x^0) \cap (y^0 - M(2\sigma)) = \{y^0\}$. Then for each $x \in U$ and each $y \in \Phi(x) \cap (y^0 - M(\sigma))$ it holds

$$\|y - y^0\| \leq \frac{L(1 + \sigma)}{\sigma} \|x - x^0\|.$$  

Our aim is to obtain optimality conditions for svp (1) in terms of Dini derivatives. For the svf $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^k$ the Dini derivative of $\Phi$ at $(x^0, y^0)$, $y^0 \in \Phi(x^0)$, in direction $u \in \mathbb{R}^n$ is defined as the upper limit

$$\Phi'(x^0, y^0; u) = \operatorname{Limsup}_{t \to 0^+} \frac{1}{t} (\Phi(x^0 + tu) - y^0).$$

### 3 First-order optimality conditions

**Theorem 1 (Necessary Conditions, w-minimizers)** Consider svp (1) with $C \subset \mathbb{R}^m$ and $K \subset \mathbb{R}^p$ closed convex cones, and $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $G : \mathbb{R}^n \rightarrow \mathbb{R}^p$ svf. Let the pair $(x^0, y^0)$, $x^0 \in \mathbb{R}^n$, $y^0 \in F(x^0)$, be a w-minimizer of sep (1), and let $z^0 \in G(x^0) \cap (-K)$. Then

$$\forall u \in \mathbb{R}^m : (F \times G)'(x^0, (y^0, z^0); u) \cap (-\operatorname{int} C \times \operatorname{int} K[-z^0]) = \emptyset. \quad (2)$$

**Proof.** Suppose the contrary, that there exists $(\tilde{y}^0, \tilde{z}^0) \in (F \times G)'(x^0, (y^0, z^0); u^0)$, such that $\tilde{y}^0 \in \operatorname{int} C$, $\tilde{z}^0 \in \operatorname{int} K[-z^0]$. Let $\tilde{y}^0 = \lim_k (1/t_k)(y^k - y^0)$, $\tilde{z}^0 = \lim_k (1/t_k)(z^k - z^0)$, where $y^k \in F(x^0 + t_k u^0)$ and $z^k \in G(x^0 + t_k u^0)$ for some $t_k \to 0^+$ and $u^0 \in \mathbb{R}^n$. These equalities imply that $y^k \to y^0$ and $z^k \to z^0$, and the boundedness of the sequences $\{y^k\}$ and $\{z^k\}$.

Let $\bar{\eta} \in K'$, $\|\bar{\eta}\| = 1$. We show that there exists a positive integer $k(\bar{\eta})$ and a neighbourhood $V(\bar{\eta})$ of $\bar{\eta}$, such that $\langle \eta, z^k \rangle < 0$ for $k > k(\bar{\eta})$ and $\eta \in V(\bar{\eta})$. For this purpose we consider the following cases:

1. $\bar{\eta} \in K' \setminus [-z^0]$. Since $\tilde{z}^0 \in \operatorname{int} K[-z^0]$, we have

$$\lim_k \frac{1}{t_k} \langle \bar{\eta}, z^k - \tilde{z}^0 \rangle = \langle \bar{\eta}, \tilde{z}^0 \rangle < 0.$$  

Therefore there exists $k(\bar{\eta})$, such that for all $k > k(\bar{\eta})$ it holds $\langle \bar{\eta}, z^k \rangle < \langle \bar{\eta}, z^0 \rangle = 0$. Now the boundedness of the sequence $\{z^k\}$ implies the existence of a neighbourhood $V(\bar{\eta})$ of $\bar{\eta}$, such that $\langle \eta, z^k \rangle < 0$ for all $k > k(\bar{\eta})$.

2. $\bar{\eta} \in K' \setminus K'[-z^0]$. We have $\langle \bar{\eta}, \tilde{z}^0 \rangle < 0$, whence $\langle \bar{\eta}, z^k \rangle < 0$ for all $k > k(\bar{\eta})$ with suitable $k(\bar{\eta})$. This implies as above $\langle \eta, z^k \rangle < 0$ for all $k > k(\bar{\eta})$ and $\eta \in V(\bar{\eta})$ with suitable neighbourhood $V(\bar{\eta})$ of $\bar{\eta}$.

The set $\Gamma = \{\eta \in K' \mid \|\eta\| = 1\}$ is compact, whence $\Gamma \subset V(\bar{\eta}^1) \cup \ldots \cup V(\bar{\eta}^s)$ for some $\bar{\eta}^1, \ldots, \bar{\eta}^s \in \Gamma$. Let $k_0 = \max(k(\bar{\eta}^1), \ldots, k(\bar{\eta}^s))$. Take $k > k_0$. Then $\langle \eta, z^k \rangle < 0$ for all $\eta \in \Gamma$, and hence for all $\eta \in K'$. Therefore $z^k \in \operatorname{int} K \subset -K$. In other words, the points $x^0 + t_k u^0$ are feasible.
According to the made assumption \( \bar{y}^0 = \lim_k (1/t_k)(y^k - y^0) \in -\text{int } C \). Therefore \( y^k - y^0 \in -\text{int } C \) for all sufficiently large \( k \), a contradiction to the hypothesis that \( (x^0, y^0) \) is a \( w^- \)-minimizer of (1). \( \square \)

**Remark 1** Condition (2), which can be referred to as primal form condition, can be substituted by the equivalent dual form condition

\[
\forall u \in \mathbb{R}^n \setminus \{0\}, \forall (\bar{y}^0, \bar{z}^0) \in (F \times G)'(x^0, (y^0, z^0); u) : \\
\exists (\xi, \eta) \in C' \times K'[\bar{z}^0], (\xi, \eta) \neq (0, 0) : \langle \xi, \bar{y}^0 \rangle + \langle \eta, \bar{z}^0 \rangle \geq 0.
\] (3)

**Theorem 2 (Sufficient Conditions, \( p^- \)-minimizers)** Consider sup (1) with \( C \subset \mathbb{R}^m \) pointed closed convex cone, \( K \subset \mathbb{R}^p \) closed convex cone, \( F : \mathbb{R}^n \rightharpoonup \mathbb{R}^m \) locally \( C \)-Lipschitz svf, and \( G : \mathbb{R}^n \rightharpoonup \mathbb{R}^p \) locally Lipschitz svf. Suppose that the pair \( (x^0, y^0) \), \( x^0 \in \mathbb{R}^n \), \( y^0 \in F(x^0) \), is such that \( y^0 \in \text{p-Min}_C F(x^0) \), and there exists \( z^0 \in G(x^0) \) for which

\[
\forall u \in \mathbb{R}^n \setminus \{0\} : (F \times G)'(x^0, (y^0, z^0); u) \cap (- (C \times K[-z^0])) = \emptyset.
\] (4)

Suppose also that the svf \( G \) satisfies the following condition:

\[
\mathcal{G}(x^0, z^0) : \\
\exists U \in \mathcal{N}(x^0) : \exists \ell > 0 : \forall x \in U : \\
G(x) \cap (-K) \neq \emptyset \Rightarrow G(x) \cap \ell \|x - x^0\| \bar{B}_p(z^0) \cap (-K) \neq \emptyset.
\]

Then \( (x^0, y^0) \) is an \( i^- \)-minimizer of sup (1).

**Proof.** We can assume without loss of generality that \( \mathbb{R}^m \) (the image space of \( F \)) is supplied with an Euclidean norm, with respect to which the cone \( C \) is non-obtuse. We may assume that \( F \) is \( C \)-Lipschitz with constant \( L > 0 \) on \( \bar{B}_n(x^0) \). Suppose that \( (x^0, y^0) \) is not an \( i^- \)-minimizer. Fix a sequence \( \varepsilon_k \to 0^+ \). According to the assumption, there exist sequences \( t_k \to 0^+ \), and \( u^k \in \mathbb{R}^n \), \( \|u^k\| = 1 \), such that:

1. \( G(x^0 + t_k u^k) \cap (-K) = \emptyset \),
2. \( D(F(x^0 + t_k u^k) - y^0) \cap -C) < \varepsilon_k t_k \).

Passing to a subsequence, we may assume that \( u^k \to u^0 \), and \( 0 < t_k < r \).

By the \( C \)-Lipschitz property of \( F \) we have

\[
\frac{1}{t_k} \left((x^0 + t_k u^0) - y^0\right), -C) < \varepsilon_k + L\|u^k - u^0\|.
\]

Let \( y^k \in F(x^0 + t_k u^0) \) be such that \( D(\bar{y}^k, -C) < \varepsilon_k + L\|u^k - u^0\| \), where \( \bar{y}^k = (1/t_k)(y^k - y^0) \). The sequence \( \{\bar{y}^k\} \) is bounded, which follows from the following reasoning. Since \( y^0 \in p\text{-Min}_C F(x^0) \), there exists \( \sigma, 0 < \sigma < 1/2 \), such that \( F(x^0) \cap (y^0 - C(2\sigma)) = \{y^0\} \). Let \( k \) be such that \( \varepsilon_k + L\|u^k - u^0\| < L \), whence \( D(y^k - y^0, -C) < \varepsilon_k t_k \). Then it holds \( \|\bar{y}^k\| < L (1 + 1/\sigma) \). Indeed, assume on the contrary, that \( \|\bar{y}^k\| > L (1 + 1/\sigma) \), or equivalently \( \|y^k - y^0\| > L (1 + 1/\sigma) \). We have

\[
D(y^k - y^0, -C) < \varepsilon_k t_k < \sigma \varepsilon_k t_k < \sigma \|y^k - y^0\|.
\]

This inequality shows that \( y^k - y^0 \in -C(\sigma) \), whence, from Lemma 2 we get

\[
\|y^k - y^0\| \leq \frac{L (1 + \sigma)}{\sigma} \|x^0 + t_k u^0 - x^0\| = L \left(1 + \frac{1}{\sigma}\right) t_k,
\]

4
a contradiction.

We proved that the sequence \( \{ \tilde{g}^k \} \) is bounded and \( \| \tilde{g}^k \| \leq L (1 + 1/\sigma) \) for all sufficiently large \( k \). Passing to a subsequence, we may assume that \( \tilde{g}^k \to \tilde{y}^0 \), whence \( \| \tilde{y}^0 \| \leq L (1 + 1/\sigma) \) and \( \tilde{y}^0 \in F'(x^0, y^0; u^0) \). Taking a limit in the inequality \( D(\tilde{g}^k, -C) < \varepsilon_k + L \| u^k - u^0 \| \) we get \( D(\tilde{y}^0, -C) \leq 0 \). Since \( C \) is closed, this inequality gives \( \tilde{y}^0 \in -C \).

The hypothesis \( G(x^0 + t_k u^k) \cap (-K) = \emptyset \) give that \( G_0(x^0 + t_k u^k) \cap (-K) \neq \emptyset \), where \( G_0(x) = G(x) \cap \ell \| x - x^0 \| B_p(z^0) \). The local Lipschitz property of \( G \) gives that there exists a point \( z^k \in G(x^0 + t_k u^0) \) such that \( D(z^k, G_0(x^0 + t_k u^k)) \leq L t_k \| u^k - u^0 \| \) (here we suppose that \( G \) is locally Lipschitz with constant \( L \) on \( r B_n \)). From the triangle inequality we get \( \| z^k - z^0 \| \leq (\ell + L \| u^k - u^0 \|) t_k \).

Putting \( \bar{z}^k = (1/t_k)(z^k - z^0) \), we have \( \| \bar{z}^k \| \leq (\ell + L \| u^k - u^0 \|) \leq \ell + 2L \). Therefore the sequence \( \bar{z}^k \) is bounded. Passing to a subsequence we may assume that \( \bar{z}^k \to \bar{z} \).

The construction of \( z^k \) yields the existence of \( \tilde{z}^k \in G_0(x^0 + t_k u^k) \cap (-K) \), such that \( z^k \in \tilde{z}^k + L t_k \| u^k - u^0 \| B_p \), whence for arbitrary \( \eta \in K'[-z^0] \), \( \| \eta \| = 1 \), we have

\[
\langle \eta, \bar{z}^k \rangle = \frac{1}{t_k} \langle \eta, z^k \rangle \leq \frac{1}{t_k} \langle \eta, \bar{z}^k \rangle + L \| u^k - u^0 \| \leq L \| u^k - u^0 \|.
\]

Here we have used \( \langle \eta, \bar{z}^k \rangle \leq 0 \), a consequence of \( \bar{z}^k \in -K \). Taking the limit in the above inequality, we get \( \langle \eta, z^0 \rangle \leq 0 \), whence

\[
D(\tilde{z}^0, -K[-z^0]) = \sup \{ \langle \eta, z^0 \rangle \mid \eta \in K'[-z^0], \| \eta \| = 1 \} \leq 0.
\]

Regarding that \( K'[-z^0] \) is closed, this gives \( z^0 \in -K[-z^0] \).

We have used the same sequence \( t_k \to 0^+ \) to construct both \( \tilde{y}^0 \) and \( \bar{z} \), hence we have \((\tilde{y}^0, \bar{z}) \in (F \times G)'(x^0, (y^0, z^0); u^0) \). So far we have proved that \((\tilde{y}^0, \bar{z}) \in -(C \times K[-z^0]) \).

On the other hand condition (4) gives \((\tilde{y}^0, \bar{z}) \notin -(C \times K[-z^0]) \), a contradiction. \( \square \)

**Remark 2** Condition (4), which can be referred to as primal form condition, can be substituted by the equivalent dual form condition

\[
\forall u \in \mathbb{R}^n \setminus \{0\} : \forall (\tilde{y}^0, \bar{z}) \in (F \times G)'(x^0, (y^0, z^0); u) : \\
\exists (\xi, \eta) \in C^* \times K'[-z^0], (\xi, \eta) \neq (0, 0) : \langle \xi, \tilde{y}^0 \rangle + \langle \eta, \bar{z} \rangle > 0.
\]

(5)

The next example shows that without condition \( G(x^0, x^0) \) Theorem 2 is not true.

**Example 1** Consider problem (1) with \( n = 1, m = 1, p = 2, C = \mathbb{R}_+, K = \mathbb{R}^2_+, F : \mathbb{R} \to \mathbb{R} \) arbitrary single-valued differentiable function, and \( G : \mathbb{R} \to \mathbb{R}^2 \) given by \( G(x) = [(|x|, -1), (-|x|, 0)] \). Let \( x^0 = 0, y^0 = F(x^0), z^0 = (0, -1) \). All conditions of Theorem 2, with exception of \( G(x^0, z^0) \), are satisfied, independently on the concrete function \( F \).

In particular \( K[-z^0] = \mathbb{R}_+ \times \mathbb{R} \) and \( (F \times G)'(x^0, (y^0, z^0); u) = (F'(0) u, G'(x^0, z^0; u)) \), where \( G'(x^0, z^0; u) = \{ |u| \} \times \mathbb{R}_+ \), which verifies condition (4). Since any point \( x \in \mathbb{R} \) is feasible, problem (1) is equivalent to the optimization problem \( \min F(x), x \in \mathbb{R} \). But, if for instance \( F(x) = -x^2 \), the point \( x^0 \) is not an i-minimizer.

The following theorem is a modification of Theorem 2 and is proved similarly.
Theorem 3 Consider svp (1) with $C \subset \mathbb{R}^m$ and $K \subset \mathbb{R}^p$ pointed closed convex cones, and $F : \mathbb{R}^n \rightharpoonup \mathbb{R}^m$ and $G : \mathbb{R}^n \rightharpoonup \mathbb{R}^p$ respectively locally C-Lipschitz and locally K-Lipschitz svf. Suppose that the pair $(x^0, y^0)$, $x^0 \in \mathbb{R}^n$, $y^0 \in F(x^0)$, is such that $y^0 \in p\text{-}\text{Min}_C F(x^0)$, and there exists $z^0 \in G(x^0)$ for which $z^0 \in p\text{-}\text{Min}_K G(x^0)$ and condition (4) holds. Suppose also that the svf $G$ satisfies condition $G(x^0, z^0)$. Then $(x^0, y^0)$ is an $i$-minimizer of svp (1).

When the functions $F$ and $G$ are single-valued, then problem (1) transforms into the vector optimization problem min$_{C F(x), G(x)} \in -K$, and Theorems 1 and 2 reduce to those proved in [3]. Then the conditions $y^0 \in p\text{-}\text{Min}_C F(x^0)$ and $G(x^0, z^0)$ are automatically satisfied.

Though condition $G(x^0, z^0)$ does not appear in Theorem 1, the interesting applications of this theorem could be those in which $G(x) \cap (-K)$ possess points near $z^0$. Indeed, suppose that $\forall \ell > 0 : \exists U \in N(x^0) : \forall x \in U : G(x) \cap (-K) \neq \emptyset \Rightarrow G(x) \cap \ell \|x - x^0\| \bar{B}_p(x^0) \cap (-K) = \emptyset$. Then $(F \times G)'(x^0, (y^0, z^0); u) = \emptyset$ for all $u \in \mathbb{R}^n$, and condition (2) is satisfied for arbitrary svf $F$.

References


