A vector variational principle

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Let $K \subset Y$ be a closed convex cone in a locally convex space Y (i.e. $K+K \subset K$ and $\alpha K \subset K$ for all $\alpha \in [0,\infty)$) and let $f : X \longrightarrow Y$ be a mapping defined on a complete metric space X. Let $D \subset K$ be a closed convex bounded subset of K such that $0 \notin \mathrm{cl}(D+K)$. We show that under mild assumptions on f and D there is $\bar{x} \in X$ such that

$$\Big(f(\bar{x}) - K\Big) \cap \Big(f(z) + d(z,\bar{x})D\Big) = \emptyset \text{ for } z \in X \setminus \{\bar{x}\}. \tag{1}$$

If $Y = \mathbb{R}$, $K = [0,\infty)$, $D = \{\epsilon\}$, $\epsilon > 0$, then (1) takes the form

$$f(z) + \epsilon d(z,\bar{x}) > f(\bar{x}) \text{ whenever } z \neq \bar{x}.$$

This is the variational inequality from Ekeland's Variational Principle (EVP) [3, 12, 13]. Thus (1) can be regarded as an extension of EVP to vector-valued mappings.

EVP is a powerful tool with many applications in optimization, control theory, subdifferential calculus, nonlinear analysis, global analysis and mathematical economy. Therefore, several formulations of EVP for vector-valued and set-valued mappings are proved e.g. in [1, 2, 4, 5, 9, 10, 11, 14].

However, the common feature of those formulations is their 'directional' character, more precisely, instead of (1) it is proved that

$$\Big(f(\bar{x}) - K\Big) \cap \Big(f(z) + d(z,\bar{x})k_0\Big) = \emptyset \text{ for } z \in X \setminus \{\bar{x}\}, \tag{2}$$

with $0 \neq k_0$ chosen from the ordering cone K. Under additional assumptions on cone K, Németh (cf. Theorem 6.1 of [11]) proved (2) with $d(z,\bar{x})k_0$ replaced by $r(z,\bar{x})$, where $r : X \times X \to K$ is a mapping such that: (i) $r(u,v) = 0 \Leftrightarrow u = v$, (ii) $r(u,v) = r(v,u)$, (iii) $r(u,z) \in r(u,v) + r(v,z) - K$ for any $u,v,z \in X$.

As in the case of scalar-valued mappings, the validity of EVP for vector- or set-valued mappings is usually verified on the basis of topological arguments and the core of the proofs is Cantor's intersection theorem. In contrast to that, we prove EVP for vector-valued mappings by combining topological and set-theoretic methods. The main set-theoretic tool is Theorem 3.7 of [8] providing sufficient conditions for the existence of maximal elements of countably orderable sets [8, Definition 2.1]. The application of set-theoretic methods allows us to prove (1) which reduces to (2) when $D = \{k_0\}$, $0 \neq k_0 \in K$.

The talk is based on the common paper with Dariusz Zagrodny which is published in Archiv der Mathematik (vol. 93, 2009, no. 6).

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