Ivan Moscati

How cardinal utility entered economic analysis during the Ordinal Revolution

2012/5

UNIVERSITÀ DELL'INSUBRIA
FACOLTÀ DI ECONOMIA

http://eco.uninsubria.it

In questi quaderni vengono pubblicati i lavori dei docenti della Facoltà di Economia dell'Università dell'Insubria. La pubblicazione di contributi di altri studiosi, che abbiano un rapporto didattico o scientifico stabile con la Facoltà, può essere proposta da un professore della Facoltà, dopo che il contributo sia stato discusso pubblicamente. Il nome del proponente è riportato in nota all'articolo. I punti di vista espressi nei quaderni della Facoltà di Economia riflettono unicamente le opinioni degli autori, e non rispecchiano necessariamente quelli della Facoltà di Economia dell'Università dell'Insubria.

These Working papers collect the work of the Faculty of Economics of the University of Insubria. The publication of work by other Authors can be proposed by a member of the Faculty, provided that the paper has been presented in public. The name of the proposer is reported in a footnote. The views expressed in the Working papers reflect the opinions of the Authors only, and not necessarily the ones of the Economics Faculty of the University of Insubria.

Printed in Italy in **december** 2012
Università degli Studi dell'Insubria
Via Monte Generoso, 71, 21100 Varese, Italy

How cardinal utility entered economic analysis during the Ordinal Revolution*

Ivan Moscati
University of Insubria, Varese (Italy)
This draft: November 2012

* I am grateful to the participants at the 2012 meetings of the ESHET, the HES and the STOREP for their helpful comments on previous drafts of the paper. I also thank the members of the staff of the Rare Book & Manuscript Library at Duke University for their help when I was working in the Paul Samuelson archive, and the Center for the History of Political Economy for its hospitality during my stay at Duke.

Abstract

The paper shows that cardinal utility entered economic analysis during the Ordinal Revolution initiated by Pareto and not, as many popular histories of utility theory assume, before it. Cardinal utility was the outcome of a discussion begun by Pareto about the capacity of ranking transitions among different combinations of goods. The discussion simmered away during the 1920s and early 1930s, underwent a decisive rise in temperature between 1934 and 1938, and continued with some final sparks until 1944. The paper illustrates the methodological and analytical issues and the measurement-theoretic problems, as well as the personal and institutional aspects that characterized this debate. Many eminent economists of the period contributed to it, with Samuelson in particular playing a pivotal role in defining and popularizing cardinal utility. Based on archival research in Samuelson's papers at Duke University, the paper also addresses an issue of priority associated with the mathematical characterization of cardinal utility.

Keywords

Cardinal utility; Ordinal Revolution; Utility analysis; Utility measurement; Samuelson.

JEL Classification

B13 (History of Economic Thought through 1925, Neoclassical);
B21 (History of Economic Thought since 1925, Microeconomics);
B40 (Economic Methodology, General);
D11 (Consumer Economics, Theory).

1. Introduction

Ordinal and cardinal utility are two key concepts of current economic theory. Consumer demand analysis is almost exclusively based upon ordinal utility, which is identified by the uniqueness of the utility function up to increasing transformations. Cardinal utility is more restrictive and is identified by the uniqueness of the utility function only up to *linearly* increasing transformations; it plays an important role in the theory of decisions under risk, the theory of intertemporal decisions, and welfare analysis.[1]

The consensus among historians is that William Stanley Jevons and the other early marginalists of the late nineteenth century were concerned with cardinal utility. Then, in the so called "Ordinal Revolution", which was inaugurated by Vilfredo Pareto ([1900] 2008, [1909] 1971) and concluded by John Hicks's *Value and Capital* (1939), utility theorists moved away from cardinalism and embraced an ordinal approach to utility (see e.g. Schumpeter 1954, Niehans 1990, Ingrao and Israel 1990, Mandler 1999, Giocoli 2003).

In opposition to this conventional narrative, I have argued in a companion paper (Moscati 2013) that Jevons and the other founders of marginal utility theory were not cardinalists in the current sense of the term. They were not interested in linearly increasing transformations of the utility function, or in the ranking of utility differences that can be associated with these transformations. Rather, they conceived of utility in a more restrictive way consistent with the so-called "classical" understanding of measurement, which dates back to Aristotle. According to the classical view, measuring an object (e.g. the length of a table) consists of comparing it with some other object that is taken as a unit (e.g. a meter rule) and then assessing the numerical ratio between the unit and the object to be measured. The founders of marginal utility theory applied this classical conception to the measurement of utility and, accordingly, focused on the possibility of ascertaining a unit of utility and assessing utility ratios, which is more demanding than ranking utility differences.[2]

More generally, in that companion paper I have contended that the traditional dichotomy between cardinal and ordinal utility is inadequate to tell an accurate history of utility theory. A third form of utility consistent with the classical

---

[1] Formally, if $U(x)$ is a utility function representing the preferences of an individual, in an ordinal-utility framework any transformation $F[U(x)]$ of the utility function such that $F'>0$, also represents his preferences. In a cardinal-utility framework, only linearly increasing transformations of the form $\alpha U(x)+\beta$, where $\alpha>0$, represent the individual's preferences. These transformations are also called "affine positive". See Fishburn 1970.

[2] Classical measurement is also labeled "ratio measurement". Formally, it is associated with proportional transformations of the unit of measurement (e.g., from inches to millimeters) that do not modify the zero point of measurement. Utility measurable in a classical or ratio sense is unique only up to transformations of the form $\alpha U(x)$, where $\alpha>0$.

understanding of measurement, namely classically-measurable utility, should be added to the picture if we want to appreciate the problem situation that the early marginalists were facing. These economists clearly perceived, on the one hand, that the measurability of utility would have made their economic theories scientifically sounder and more defensible against the attacks of their critics; on the other hand, however, they thought that they knew what measurement was (i.e. classical measurement), and consequently reckoned that the utility featuring in their theories was not measurable. In fact, the discussions of Jevons and other early marginalists as to the measurability of utility, or the extent to which their theories actually relied on such measurability, largely originated from the tension between their classical understanding of measurement and the fact that their scientific practices did not square with it. The traditional twofold categorization in terms of cardinal utility and ordinal utility is inadequate to illuminate properly that tension.

But if the early marginalists were not cardinalists in the current sense of the term, when did cardinal utility enter the scene? The present paper contends that cardinal utility entered economic analysis *not before, but during the Ordinal Revolution*.

Cardinal utility was the outcome of a long-lasting discussion, inaugurated by Pareto in his *Manual of Political Economy* ([1909] 1971), as to an individual's capacity to rank transitions from one combination of goods to another, and the differences between the utility indices associated with those transitions. During the 1920s and early 1930s, however, these rankings were not associated with linearly increasing transformations of the utility function or with the expression "cardinal utility". The beginning of the conclusive phase of the Ordinal Revolution in 1934 prompted a sudden and decisive acceleration toward the definition of cardinal utility. In particular, in 1934 Oskar Lange connected the ranking of transitions with utility unique up to linearly increasing transformations; in 1936 Franz Alt showed under what exact assumptions Lange's connection is valid; and in 1938 Paul Samuelson coupled the expression "cardinal utility" with utility unique up to linearly increasing transformations. Between 1938 and 1944, this specific meaning of cardinal utility stabilized, especially thanks to its usage by Samuelson. By showing how cardinal utility entered economic analysis during the Ordinal Revolution, the paper not only revises the conventional narrative of the history of cardinal utility, but also adds to a series of recent studies that have enriched and partially modified the standard picture of the Ordinal Revolution itself (Lenfant 2006, 2012; Hands 2010, 2011).

The history of the progressive emergence of cardinal utility has many dimensions, which structure the narrative of the paper. Firstly, different stances on the ranking of transitions reflected the different methodological views regarding demand analysis that confronted each other during the Ordinal

Revolution. At one extreme, some economists saw the ranking of transitions as a psychologically plausible and scientifically legitimate way of preserving decreasing marginal utility and other intuitive but non-ordinal notions of pre-Paretian demand analysis. At the other extreme were those who criticized ranking as empirically unverifiable and superfluous to demand theory.

Secondly, these discussions contained a major analytical dimension. Economists long identified the ranking of transitions with the ranking of utility differences, but when this identification was challenged the problem arose of defining the exact assumptions under which the utility function is unique only up to linear transformations. The paper illustrates how this analytical problem was dealt with.

Thirdly, an issue of priority is associated with this analytical problem. Alt in 1936 and Samuelson in 1938 stated conditions restricting the transformations of the utility function to the linearly increasing ones; however Samuelson did not cite Alt. Based on archival research into Samuelson's papers collected at Duke University, I show that Samuelson was aware of Alt's contribution.

Fourthly, by reconstructing Samuelson's intensive dealing with utility in the late 1930s and early 1940s, and demonstrating his pivotal role in defining and popularizing the current meaning of cardinal utility, the paper modifies a widespread image of the American economist according to which his chief goal during the Ordinal Revolution was to free economic theory from any vestige of the utility concept.

Fifthly, the paper calls attention to the personal and institutional aspects of the discussions leading to the definition of cardinal utility. Most of the economists involved knew each other, often through their participation in the Econometric Society, and were located in a limited number of institutions, such as the London School of Economics, the University of Vienna and Harvard University. Since Alt played an important role but is little known in the history of economics, I also investigate the biographical background that led him to contribute to the definition of cardinal utility.

Last but not least, the discussions leading to cardinal utility are important for the history of utility measurement. In particular, it was in the course of these discussions, particularly in the years 1934-1938, that the first clear signs of a non-classical understanding of measurement appeared in utility theory.

2. Pareto and the ranking of transitions

In his *Manual*, Pareto showed that the main results of demand and equilibrium analysis are independent of the measurability of utility, and can be drawn from the single assumption that individuals are able to rank combinations of goods. Under this assumption, utility can be expressed by indices that represent the

preference ranking of the individual in the sense that, if the individual prefers one combination of goods over another, the former must have a larger index. In the Mathematical Appendix, Pareto connected utility indices with increasing transformations, i.e., if U is a utility index representing the individual's preferences, also any $F(U)$ with $F'>0$ represents them, because $F(U)$ still expresses the individual's preference order.

A theory based on ordinal utility indices implies the dismissal of earlier notions of utility theory that are not invariant to increasing transformations. Among these notions are the principle of decreasing marginal utility and the definition of complementary and substitute goods based on how the marginal utility of one good varies when the quantity of another varies.[3] It is not clear whether Pareto was fully aware of these restrictive implications of the ordinal approach. At any rate, when he needed decreasing marginal utility or the traditional definition of complementarity for considerations going beyond pure equilibrium analysis, he did refer to these notions (Bruni and Guala 2001).

Throughout his scientific career, Pareto intended by measurement classical measurement (Pareto ([1892] 2007: 58-60, [1896] 1964: 8-9, [1909] 1971: 112, 396). Accordingly, for him the utility indices did not measure utility. In the *Manual*, however, he suggested that we can imagine at least two special cases in which utility is measurable and utility indices can be interpreted as utility measures.

The first case, which is peripheral to our story, is when the marginal utility of each commodity depends only on the quantity of that commodity. Following a suggestion by Irving Fisher (1892), Pareto (395–396) showed that, in this situation, the indices expressing marginal utilities are unique up to proportional transformations, and the marginal utility of any commodity could be taken as the unit to measure the marginal utility of other commodities in the classical sense (see Montesano 2006). But Pareto discarded the hypothesis that the utilities of different commodities are independent, and therefore also the idea that utility could be measured in the way suggested by Fisher. Following Pareto, after 1910 most utility theorists ruled out the hypothesis of independent marginal utilities, and therefore the idea of measuring utility using that hypothesis.[4]

---

[3] Let $U(x_1, x_2, \ldots, x_n)$ be the utility function, and denote $U_{ij}=\partial^2 U/\partial x_i \partial x_j$. The principle of decreasing marginal utility implies that $U_{ii}<0$. However, the sign of U_{ii} is not invariant to increasing transformations of U. According to the definition used by early marginalists, two goods i and j are complementary if $U_{ij}>0$, and substitute if $U_{ij}<0$. However, not even the sign of U_{ij} is invariant to increasing transformations of U.

[4] The main exceptions were Fisher himself (1927) and Ragnar Frisch ([1926] 1971, 1932). They proposed statistical methods to measure the marginal utility of money that relied on the assumption that, for each individual, the marginal utilities of commodities are independent, and on the additional assumption that different individuals have identical utility functions. However, in the context of post-Paretian utility theory both assumptions appeared highly problematic and

The second case is for us the more relevant. For Pareto, utility is measurable when individuals: (1) are not only able to rank consumption alternatives, but (2) are also capable of ranking transitions from one alternative to another, and (3) are even capable of stating that a given transition is equally or twice as preferable as another.

For Pareto, assumption 2 was in accord with the idea of decreasing marginal utility and appeared plausible, at least for adjacent transitions. In particular, he claimed that this assumption restricts the arbitrariness of the utility index to those increasing transformations which display the following additional property:

> If in passing from [combination] I to [combination] II the man experiences more pleasure than in passing from II to III, the difference between the indices of I and II is greater than the difference between the indices of II and III. (192)

However, Pareto did not provide a mathematical characterization of how this property restricts the set of admissible transformations of utility indices. In particular, he did not associate assumption 2 to linearly increasing transformation of the indices. It should also be noted that, in the passage quoted above, Pareto takes for granted that the ranking of transitions from one combination to another implies the ranking of the differences between the utility indices associated with the combinations. As became clear much later, this implicit supposition is unwarranted.

With respect to assumption 3, Pareto discarded it as highly unrealistic and so, in the end, for him utility remained immeasurable.

3. The ranking of transitions in the 1920s and early 1930s

In the 1920s and early 1930s, Pareto's discussion about the ranking of transitions and utility differences was picked up by a number of eminent economists. We focus here on Luigi Amoroso, Arthur Bowley, Paul Rosenstein-Rodan and Oskar Morgenstern because their discussions of transition rankings were mentioned in the debate of the mid-1930s.[5]

Amoroso (1886-1965) was probably the leader of the Pareto School in interwar Italy and a founding fellow of the Econometric Society. In his *Lezioni di economia matematica* (Lectures on mathematical economics), Amoroso (1921: 91-92) endorsed Pareto's idea that individuals are capable of ranking transitions from one combination to another (i.e. Pareto's claim 2), and argued that this capacity

were criticized from many quarters. At any rate, Fisher's and Frisch's statistical approaches to utility measurement had no influence on the discussions that led to the definition of cardinal utility.

[5] Among the others who touched upon the ranking of transitions are de Pietri-Tonelli (1927) and Frisch ([1926] 1971).

restricts the arbitrariness of the utility index in a way that makes the comparison of utility differences meaningful. This, in turn, allows preservation of the principle of decreasing marginal utility and the traditional definition of complementarity without returning to the pre-Paretian view that utility is measurable. Like Pareto, Amoroso rejected the hypothesis that individuals are capable of stating how many times one transition is larger than another (i.e., Pareto's point 3).

Bowley (1869-1957) was a Cambridge-trained economist and statistician, who spent most of his academic career at the LSE. A founding member of the Econometric Society, Bowley would become a senior affiliate of the group of young economists who formed around Lionel Robbins after his 1929 appointment to the LSE chair in economics. The presentation of utility theory in Bowley's 1924 *The Mathematical Groundwork of Economics* was the most sympathetic to Pareto's ordinal approach of any work published in England in the 1920s. In the introduction, Bowley (1924: 1-2) made clear that, although all major results of utility analysis can be based on the assumption that individuals are able to rank combinations of goods, some economic propositions require that individuals are also able to rank transitions from one combination to another. These propositions are the ones involving the principle of decreasing marginal utility or the traditional definition of complementarity.

Rosenstein-Rodan (1902-1985) and Morgenstern (1902-1977), then two young members of the Austrian School, also discussed the ranking of transitions and utility differences. Both had studied in Vienna under Hans Mayer, and in 1929 they became managing editors of the *Zeitschrift für Nationalökonomie*. In the late 1920s, both were awarded fellowships by the Rockefeller Foundation which allowed them, among other things, to visit the LSE. In 1930, Rosenstein-Rodan accepted a position of lecturer at University College London, and he quickly became an active member of Robbins' group at the LSE. Morgenstern left Vienna for the United States in early 1938 and ended up in Princeton (Howson 2011, Leonard 2010, Schmidt and Weber 2012).

In 1927, Rosenstein-Rodan published an important entry on "Marginal Utility" in the German encyclopedia *Handwörterbuch der Staatswissenschaften* in which he argued ([1927] 1960: 75) that "utilities are not susceptible of exact quantitative measurement". In accord with Pareto, however, he added that utility measurement is not necessary for economic calculation, since "all we have to do is to decide whether one utility is larger or smaller than the other." (75) Rosentein-Rodan admitted the possibility that individuals are also able to rank utility differences but, like Pareto and Amoroso, he denied that individuals are capable of stating how much larger or smaller the utility difference is.

In 1931, Morgenstern published in the German journal *Schriften des Vereins für Sozialpolitik* an article on "Die drei Grundtypen der Theorie des subjektiven

Wertes" (The three fundamental types of the theory of subjective value). Here Morgenstern (1931: 13-14) argued that economic subjects are able to compare not only utilities but also utility differences, and that these two abilities are all the subjects need for behaving rationally in the economy. For Morgenstern, an exact measurement of utility is not only unnecessary for economics and economic activity but also impossible because utilities are qualities rather than quantities, and as such cannot be measured.

To sum-up: under Pareto's influence, in the 1920s and early 1930s a number of economists came to see utility as an ordinal index that is not measurable in the classical sense. Some of them viewed the ranking of transitions as a way of preserving the principle of decreasing marginal utility and the traditional definition of complementary within the boundaries of the ordinal framework set by Pareto. However, the exact meaning and implications of this ranking remained unexplored.

Notably, none of the economists discussing transition rankings employed the expression "cardinal utility". In effect, before 1934 the cardinal-ordinal terminology was almost unheard in economics. I now briefly outline how this terminology entered the field.

4. "Cardinal" and "ordinal" from mathematics to economics

The cardinal-ordinal terminology had been introduced in the late nineteenth century by German mathematicians in the course of discussions about the nature of numbers. For our purposes, it suffices to say that in these discussions "cardinal number" had two main meanings. Ernst Schröder (1873) and other mathematicians associated cardinal numbers with the counting of objects and the decomposition of a quantity into units, thus relating cardinals to the classical understanding of measurement. In his theory of transfinite sets, by contrast, Georg Cantor ([1887] 1932) referred to cardinal numbers as characterizing a family of sets whose elements can be put into a one-to-one correspondence. In neither mathematical meaning, however, did cardinal numbers have anything to do with the ranking of differences between objects (see Moscati 2013).

The terms "ordinal" and "cardinal" seem to have first appeared in economics in an 1893 article in the political-economy journal *Zeitschrift für die Gesamte Staatswissenschaft* by the German mathematician and economist Andreas Voigt (Schmidt and Weber 2008). In this paper, Voigt ([1893] 2008, 502) cursorily mentioned the mathematical distinction between ordinal and cardinal numbers. Voigt's notion of cardinal number was that associated with the decomposition of a quantity into units and, as such, was unrelated to differences between objects.

Among the very few who took notice of Voigt's paper were Francis Ysidro Edgeworth and Rosenstein-Rodan. In four articles in the *Economic Journal*, Edgeworth (1894, 1900, 1907, 1915) referred to Voigt and his distinction between cardinal and ordinal numbers, though always in a cursory way. Like Voigt, Edgeworth associated cardinal numbers with the availability of a unit and hence with classical measurement, and did so without mentioning utility differences. In his entry on "Marginal Utility", Rosenstein-Rodan ([1927] 1960: 75) did not use the cardinal-ordinal terminology but mentioned Voigt among those who, like Pareto, thought that "utilities are not measurable but can be compared."

A new impulse to the use of the cardinal-ordinal terminology came in 1934, when two junior members of the Robbins group at the LSE – John Hicks and Roy Allen – published a joint article that began the conclusive phase of the revolution in demand analysis initiated by Pareto.

5. Cardinal and ordinal utility by Hicks and Allen

Hicks (1904–1989) studied at Oxford, and joined the LSE in 1926 as a lecturer in economics, whereupon he began an intensive study of Pareto, Walras and Edgeworth. When Robbins arrived at the LSE in 1929, Hicks immediately became an enthusiast member of the Robbins circle and seminar. In the early 1930s he joined the Econometric Society and was elected a fellow in 1937. Allen (1906–1983) studied mathematics at Cambridge and entered the LSE department of statistics as Bowley's assistant in 1928. He was another early affiliate of the Robbins group and early member of the Econometric Society, becoming a fellow in 1935.

In 1933, Hicks and Allen co-wrote "A Reconsideration of the Theory of Value", which, after thorough discussion in Robbins' seminar (Hicks 1981), was published in two parts in the February and May 1934 issues of *Economica*, the senior LSE economics journal. The article had an immediate and strong impact on economists working on demand analysis and became a standard reference for subsequent discussions on utility theory.

Hicks and Allen (1934: 52) began with an enthusiastic endorsement of Pareto's "demonstration of the immeasurability of utility", labeling his approach "the 'ordinal' conception of utility" (54). However, for them, Pareto did not examine what adjustments in demand analysis are made necessary by the superseding of measurable utility. Some of his concepts, such as the principle of decreasing marginal utility and the traditional definition of complementary goods, are still "dependent upon a 'cardinal' conception of utility" (54–55). Therefore, these concepts have to be abandoned and, if possible, substituted by new notions consistent with the immeasurability of utility.

Hicks and Allen tried to carry out this project along the behaviorist approach that Allen had employed in an earlier attempt to base the theory of exchange on indifference curves alone (Allen 1932). The cornerstone of the Hicks–Allen analysis became the marginal rate of substitution, understood as a quantitative and observable entity independent of utility. Starting from the marginal rate of substitution, the two economists provided a new and utility-free definition of complementary and competitive goods. The new definition was based on the relative changes of demand for goods *Y* and *Z* determined by a variation of the price of good *X*.

Four aspects of this co-authored article are relevant for our story. First, the terms "ordinal" and "cardinal" occur in the paper only once, i.e., in the two passages at pages 54-55 quoted above.

Second, in all probability Hicks and Allen arrived at their cardinal-ordinal terminology through reading Edgeworth's collected papers (Edgeworth 1925), which include three of the four essays where Voigt and his distinction between cardinal and ordinal numbers is mentioned. Another possible source is Rosenstein-Rodan, who knew Voigt's 1893 article, participated in the Robbins seminars, and is mentioned by Hicks and Allen (1934: 74) as an important source of inspiration for their paper (Schmidt and Weber 2012).

Third, because of their behaviorist and utility-free approach, Hicks and Allen did not elaborate on the distinction between ordinal and cardinal utility and did not make clear what they meant by cardinal utility. Hicks and Allen (1934: 52) related Pareto's relapses into the cardinal conception of utility to Marshall, Walras, Edgeworth and their conception of utility as a measurable quantity. This suggests that they understood cardinal utility as measurable in the classical sense. At any rate, Hicks and Allen certainly did not associate cardinal utility with the ranking of utility differences or linear transformations of the utility functions, since in their article there is no sign of the latter.

Fourth, despite the fact that the terms "ordinal" and "cardinal" occurred only once in their paper, Hicks and Allen contributed immediately to their diffusion, at least within the Robbins group. Two other members of the group, namely Friedrich von Hayek and Frederic Benham, published separately in the November 1934 issue of *Economica* articles where the terms are used (once in each). However, neither Hayek nor Benham associated the term cardinal with the ranking of utility differences.[6]

[6] Hayek (1934: 401) claimed that Carl Menger understood the numbers he used to express the marginal utility of goods "not as cardinal but as ordinal figures." Benham (1934: 446) argued that utility and welfare preclude "objective measurement (whether in 'cardinal' or 'ordinal' numbers)."

6. Lange and the linear transformations of the utility function

The meaning and implications of the utility-difference ranking, as well as its relationships with the behaviorist and ordinalist approaches to demand analysis, were thoroughly investigated in a debate that took place after 1934, mainly in the *Review of Economic Studies*. The *Review* was the junior LSE economics journal, founded in 1933 by Ursula Webb, who belonged to the Robbins circle and in 1935 married Hicks; Abba Lerner, another brilliant member of the Robbins group; and Paul Sweezy, a Harvard graduate student who had visited the LSE in the academic year 1932-33. The debate was initiated by Lange, who was prompted to reconsider the discussions on transition rankings by Hicks's and Allen's claim that ordinal utility implies the abandonment of decreasing marginal utility and the traditional definition of complementary goods. Lange's article, published in the June 1934 issue of the *Review*, was entitled "The determinateness of the utility function" (Lange 1934a). Since the titles of many of the contributions generated by Lange's article repeated its title, the discussion he began can be labeled "the debate on the determinateness of the utility function".

Lange (1904-1965) was a Polish economist who studied at the University of Cracow, where in 1927 he became a lecturer in statistics. In 1929 he visited London and Cambridge. Lange was another early member of the Econometric Society, participating in its Paris (October 1932) and Leyden (September-October 1933) meetings. These were small gatherings in which the participants (22 in Paris and 30 in Leyden; Lutfalla 1933, Marschak 1934) became acquainted. At Leyden, with Allen, Hicks, Lerner and Webb also in attendance, Lange presented a paper (Lange 1932) that provided a mathematical analysis of cross-elasticity and which was approvingly referred to by Hicks and Allen (1934: 64).

In 1934, Lange left Poland on a two-year Rockefeller fellowship that brought him first to the LSE and then to Harvard, where he studied under Joseph Schumpeter. Then, in 1938, he returned to the United States, and in 1939 he was appointed professor at Chicago and also became a research associate of the Cowles Commission, which had just moved to Chicago (Dobb 1966).

In his article on the determinateness of the utility function, Lange summarized the discussion on the implications of the immeasurability of utility from Pareto to Hicks and Allen, and labeled the assumption that individuals are only able to rank combinations of goods "postulate 1", and the assumption that individuals are also capable of ranking transitions from one combination of goods to another "postulate 2". For Lange, none of the economists who admitted postulate 2, namely Pareto, Amoroso, Bowley, Rosenstein-Rodan and Morgenstern, seemed to have realized that it implies something that they discarded as implausible,

namely that individuals are also capable of stating how many times a given transition is preferable to another (this is Pareto's point 3):

> From the assumption that the individual is able to know *whether* one increase of utility is greater than another increase of utility the possibility of saying *how many times* this increase is greater than another one follows necessarily. (Lange 1934a: 220)

In fact, Lange reasoned, if postulate 2 holds we can vary combination III until the individual perceives the change of utility due to transition from II to III as equally preferable to the change of utility due to transition from combination I to combination II, that is, we can vary combination III until U(III)–U(II)=U(II)–U(I). Rearranging this equation, we obtain U(III)–U(I)=2[U(II)–U(I)], and thus that "the change of utility due to transition from I to III is *twice* the change of utility due to transition from I to II." (222). Therefore, Lange concluded, postulates 1 and 2 imply a return to "determinate", i.e. measurable, utility:

> The two fundamental assumptions used by Pareto and other writers of his and of the Austrian school (and by Professor Bowley) are equivalent to the assumption that utility is measurable. (223)

In making this point, Lange also ushered onto the stage the linearly increasing transformations of the utility function. In fact, he proved that postulate 2 restricts the admissible transformations of U to those of the form $F(U)=\alpha U+\beta$, where $\alpha>0$ (221). This means, noticed Lange, that different utility indices can differ only by a constant multiplier that fixes the unit of measurement, and by an additive constant that fixes the zero point of measurement.

Based on the result that postulate 2 restores the determinateness of the utility function and the measurability of utility, Lange indicated two alternative approaches to demand analysis. The first, based on postulate 1 alone, is sufficient to establish all equations of demand analysis. This approach reduces the assumptions to one, and this single assumption "can be expressed in terms of objective human *behaviour*, i.e. in terms of choice" (224). The second is based on postulates 1 and 2. Postulate 2 cannot be expressed in terms of choice behavior and, to have some insight about which transition an individual prefers, we have to rely on the individual's communication of the result of psychological introspection. For Lange, the main merits of the second approach are that it allows for a psychological interpretation of the equations of demand theory in terms of intuitive concepts such as decreasing marginal utility and that it permits a welfare analysis of economic equilibrium. For Lange, both approaches are legitimate and one should choose among them according to the goals aimed at.

Some comments on Lange's article are in order. To begin with, while the first to connect explicitly and formally the ranking of utility differences with linearly

increasing transformations of the utility function, he did not employ the expression "cardinal utility."

Second, Lange somehow overcame the classical notion of measurement and its identification with ratio assessments. It is true that for him postulate 2 implies the measurability of utility because it allows the assessment of how many times a utility variation is greater than another, that is, the assessment of the numerical ratio between utility variations. On the other hand, he deemed that utility indices differing only by unit and zero point of measurement provide a proper measure of total utility; and this despite the fact that if unit and zero point of utility are arbitrary, the numerical ratio between total utilities cannot be assessed. We may therefore conclude that Lange's article displays some signs of a non-classical understanding of measurement, which however remained unarticulated.

Third, Lange made explicit that the technical issue concerning the meaning and implications of the utility-difference ranking is intertwined with the more general methodological issue concerning the relationship between economics and psychology. In particular, he was neither a behaviorist nor a stern ordinalist, but acknowledged the fruitfulness of psychological introspection in economics.

Finally, like Pareto, Amoroso and the other economists who admitted postulate 2, Lange took for granted that the ranking of transitions from one combination to another and the ranking of utility differences are one and the same thing. Phelps Brown's comment on Lange's article showed that this is not the case.

7. Phelps Brown's distinction between transition rankings and utility differences

Like Hicks, Henry Phelps Brown (1906–1994) studied at Oxford, where he was taught by Robbins, and whom he replaced as lecturer in economics at New College when the latter moved to the LSE. He spent 1931-32 as a Rockefeller Traveling Fellow in the United States, visiting various universities including Chicago, where he studied under Henry Schultz, a statistician, mathematical economist, and admirer of Pareto. Back in Oxford, Phelps Brown focused on statistics and mathematical economics and became an active member of the Econometric Society (Hancock and Isaac 1998).

In the Society's Paris meeting of October 1932, Phelps Brown read a paper which argued that willingness to pay cannot be used as a measure of marginal utility (Phelps Brown 1934a). At the Leyden meeting he renewed acquaintanceship with Lange and met also Allen, Hicks, Lerner and Webb from London and Schultz from Chicago. Given his research interests and his acquaintance with Lange, the fact

that Phelps Brown commented on Lange's paper on the determinateness of utility is not surprising.

In his comment, Phelps Brown (1934b) argued that, contrary to Lange's claim, postulate 2 does not imply the measurability of utility. He began by stating that by measurement he intended the assessment of ratios between magnitudes, i.e., classical measurement, and that ordering is not measuring:

> [Measurability] consists in the possibility of expressing one magnitude as a multiple of another. It may be possible to arrange terms in an order […] but we have not reached measurement, until we can express one as a multiple of another. (66)

Phelps Brown noticed that both postulates 1 and 2 concern preference order. Postulate 1 refers to preference order over combinations of goods and allows for the introduction of a numerical index U that assigns larger numbers to more preferred combinations. Postulate 2 refers to the preference order over transitions from one combination to another, and allows for the introduction of another index, let us call it G, that assigns larger numbers to more preferred transitions. However, Phelps Brown stressed, the numbers associated by G to transitions need not be equal to the differences between the numbers associated by U to combinations. Moreover, since the numbers associated by G to transitions have only an ordinal meaning, it does not make sense to sum them. Therefore, we cannot obtain the numbers associated by U to a certain combination (e.g. combination III) by summing the numbers associated by G to the transitions that led to that combination (e.g., transitions from I to II, and from II to III):

> The first series [the series of the U-numbers] cannot be reached by integration from the second [the series of the G-numbers]: for if in the second we mark the transition from combination I to II by 6, and that from II to III by 7, the transition from I to III is not necessarily marked by 13; 6+7 has here no right to meaning. (67-68)

For Phelps Brown, Lange's argument for the measurability of utility was based on the fallacious supposition that the G-numbers can be meaningfully summed. If an individual considers transition from I to II equally preferable to transition from II to III, then G will assign to both the same number K. But, contrary to Lange, the number assigned by G to the transition from I to III need not be K+K=2K. According to Phelps Brown, Lange's fallacious inference depended on the fact that, in representing preference orders by numbers, one tends illegitimately to extend the additive properties of numbers to preferences. If we avoid numbers and, for instance, represent preferences orders by the order of words, the temptation to sum what cannot be summed disappears:

> The two included transitions [from I to II, and from II to III] are indistinguishable, and to each will therefore correspond the same term […], *maison*. We have then no temptation to suppose that if the consumer makes the transition represented by *maison* once and then once again, he has made in all a transition to be represented by 2(*maison*). (68)

Since the G-numbers cannot be summed and do not coincide with the differences between the U-numbers, then, concluded Phelps Brown, postulate 2 does not permit the expression of one utility variation as a multiple of another, i.e., utility cannot be measured.

More generally, postulate 2 does not warrant the interpretation between differences of U-numbers as marginal utilities: "It is therefore not permissible to take the first derivative of the first function [U] as giving the series appropriate to marginal utilities." (68) Thus, even if one assumes postulate 2, it is not legitimate to talk of decreasing marginal utility or to employ the traditional definition of complementarity. Also, Lange's proof that postulate 2 restricts the admissible transformations of U to those of the form $F(U)=\alpha U+\beta$ relies on the illegitimate identification of G-numbers with the differences between U-numbers.

Phelps Brown article was followed by a note by Lange (1934b) in which he refined his proof that the comparability of differences between the U-numbers restricts the admissible transformations of U to those of the form $F(U)=\alpha U+\beta$, but did not address Phelps Brown's point that postulate 2 does not warrant the comparability of the differences between the U-numbers. It appears that Lange wrote his note before reading Phelps Brown's comment. Nevertheless, in a letter sent to Samuelson on 10 May 1938, Lange acknowledged that Phelps Brown's objections were correct (on this letter, see Section 10.3).

Two brief final comments on Phelps Brown's piece are in order. First, Phelps Brown did not use the cardinal-ordinal terminology. Second, he did not investigate what assumptions should be added to postulates 1 and 2 to make sense of the sum of the G-numbers or warrant that the G-numbers coincide with the differences between U-numbers.

8. Other reactions: Bernardelli and Allen

8.1. Bernardelli's defense of postulate 2

Phelps Brown's comment on Lange's article was followed by another note by Harro Bernardelli (1906-1981). A Viennese of Italian extraction, Bernardelli had studied economics in Bonn and Frankfurt before moving to the LSE in 1933, where he entered the Robbins circle. Following a research fellowship at Liverpool, in 1937 he moved eastward, to universities in, first, Burma (Rangoon) and then New Zealand (Otago) (Donoghue 2007).

In his comment, Bernardelli accepted Lange's claim that postulate 2 implies the measurability of utility, and defended the psychological plausibility and scientific legitimacy of the postulate. Opposing both a strict ordinalism admitting only postulate 1 and Hicks's and Allen's behavioristic approach, Bernardelli argued that postulate 2 should be retained as a fundamental pillar of economic analysis, its rejection entailing "the relinquishing of many propositions which until now have been considered as undoubtedly belonging to the body of Economic Theory" (Bernardelli 1934: 71), such as the principle of decreasing marginal utility and the traditional and intuitive definition of complementarity. For Bernardelli, the theories of Pareto and Hicks–Allen are "axiomatic experiments" showing how much of our economic knowledge is independent of the second postulate, and they resemble "the behaviour of a man who cuts off one of his legs, in order to see how he gets on as a cripple." (71) But such amputation is not necessary:

> It is extraordinary how one can get on without the leg of the second postulate, as the results of Pareto, and more recently of Allen and Hicks, prove. Yet this would seem insufficient reason for making a virtue of such an amputation. (71-72)

Notably, in his comment, Bernardelli did not employ the expression "cardinal utility".

8.2. Allen's criticism of postulate 2

With a brief note in the February 1935 issue of the *Review*, Allen also entered the fray, denying the usefulness of postulate 2. Allen (1935: 155-156) argued that, since the theory of value can be developed on the basis of postulate 1 alone and postulate 2 "cannot be expressed in terms of the individual's acts of choice", it would be futile to complicate the analysis with postulate 2 unless it "works its passage". Notably, in discussing the usefulness of postulate 2, Allen took into account Phelps Brown's criticism of Lange and accordingly identified the postulate with the capacity of ranking utility increments rather than transitions.

In opposition to Lange and Bernardelli, who had argued that postulate 2 is necessary to understand complementarity and for welfare analysis, Allen claimed that this was not the case. The new definition of complementarity he and Hicks had proposed in their 1934 article was not only independent of postulate 2, but showed that the distinction between complementary and substitute goods "has nothing to do with utility or intensities of preference" and is rather based on "the inter-relations of individual demands under market conditions" (158). Postulate 2 does not even warrant welfare analysis, for which "additional, and far more serious, assumptions about the relations between the preference scales of *different* individuals are necessary." (158). In conclusion, for Allen postulate 2 does not work its passage and should be discarded.

In criticizing the usefulness of postulate 2, Allen nevertheless discussed what it would imply for the measurability of utility, and interestingly argued that postulate 2 would make utility measurable in the same sense as length. Following Lange, for Allen postulate 2 restricts the transformations of the utility index in such a way that only the unit and zero point of utility are arbitrary. But, Allen claimed, the unit and zero point of measurement are also arbitrary in the measurement of length. While the use of different units such as inches or millimeters makes the arbitrariness of the length unit apparent, the arbitrariness of the zero-length point is obscured by the circumstance that everybody accepts one particular zero mark. However, Allen observed, "there is no theoretical […] reason why a definite length such as 6 inches should not be taken as a zero mark". Therefore, "the measure of utility […] is in no essential way different from that of a physical magnitude such as length." (158)

Allen was well aware that any change in the arbitrary zero point of measurement would imply a modification of ratios, be they utility or length ratios. However, this did not prevent him from considering utility invariant to linear transformations and length as a measurable magnitude:

> It is clear that statements about one length or utility being a multiple of another are not essential to the notion of length or utility as a measurable concept. (158)

These statements show that Allen's conception of measurement went beyond the classical one expressed, for example, by Phelps Brown.

In conclusion, like Lange, Phelps Brown and Bernardelli, Allen did not use the expression "cardinal utility". He also did not address the main issue left open by Phelps Brown concerning the additional assumptions under which the ranking of transitions from one combination of goods to another can be identified with the ranking of utility differences. This issue found a solution in a contribution from Franz Alt, a young Viennese little known in the history economics.

9. The man who came in from mathematics: Alt's 1936 contribution

9.1. A biographical sketch

Alt (1910-2011) graduated in mathematics from Vienna in 1932 with a dissertation under Karl Menger that provided a definition of the curvature of a curve that generalized a definition previously given by Menger himself. From 1930 Alt became a stable participant in Menger's celebrated seminar, the *Mathematische Kolloquium*. As a Jew, Alt failed to obtain an academic position, but on Menger's recommendation was hired by Morgenstern as a private tutor in mathematics and aid in reading mathematical economics. Among the books they worked through was Bowley's *Mathematical Groundwork of Economics*. Through participation in the *Kolloquium* and his tutoring of Morgenstern, Alt became

interested in the mathematical aspects of economics, as testified by two articles (Alt 1935, [1936] 1971) and a number of reviews of economics books published between 1934 and 1938 in Morgenstern's *Zeitschrift für Nationalökonomie*.

Through Morgenstern, at some point between late 1934 and early 1935 Alt became involved in the debate over the determinateness of the utility function:

> I remember […] an afternoon tea at Morgenstern's house to which I was invited. I met a whole lot of people there, among others, an American called [Paul] Sweezy, a mathematical economist. […] Paul Sweezy showed me a reprint that he was carrying from a Polish mathematician, […] Oskar Lange, about measuring the value of economic commodities. (Alt and Akera 2006: 7)

Alt had been trained in the axiomatic mathematical tradition of David Hilbert, which was also the standard approach of Menger and other *Kolloquium* participants. In this tradition, one specifies a consistent set of axioms and proves that a given statement of interest can be logically deduced from them. With respect to the standards of proof accepted in mathematics, Alt found Lange's demonstration that postulates 1 and 2 imply the measurability of utility unsatisfactory, and began writing a reply letter to Lange that ended up becoming his 1936 article:

> Lange said if you had these two conditions [postulates 1 and 2] then that's sufficient to assign a number to every commodity by itself. […] I read that, and I was a very theoretical mathematician. That's not mathematics. That's not a proof, I thought. I began to write a letter to Oskar Lange […], and the letter grew to be 10 pages long. And I realized I was writing a paper. (8-9)

Originally written in English, Alt translated his paper into German and gave it to Morgenstern. It was published as "Über die Messbarkeit des Nutzens" (On the measurability of utility) in the June 1936 issue of the *Zeitschrift*.

The story of Alt's article has an American sequel. In May 1938, a few weeks after the annexation of Austria by Nazi Germany, Alt and his wife fled to New York, where Alt found a job as an econometrician. Around 1939, Alt received a letter from Schumpeter, who had read the *Zeitschrift* article and asked Alt whether he was still working on the measurability of utility. Schumpeter was possibly fishing to see whether Alt would be interested in a job at Harvard, but Alt did not catch his intention in time (Alt and Akera 2006: 9). In his *History of economic analysis*, Schumpeter (1954: 1063) acknowledged that Alt's 1936 article provided a satisfactory solution to the issue of utility measurement.

9.2. The paper's content

As mentioned, Alt's 1936 contribution to the determinateness-of-the-utility-function-debate was motivated by his criticism of Lange's supposed

demonstration that postulates 1 and 2 imply the measurability of utility. In particular, Alt agreed with Phelps Brown that the key flaw of Lange's argument consisted in the unwarranted extension of the properties of the preference order over transitions to the preference order over combinations of goods (Alt [1936] 1971: 431).

In the spirit of the axiomatic method, Alt added to Lange's two postulates five additional postulates concerning the properties of the preference order over transitions, the properties of the preference order over combinations, and the relationships between the two preference orders.[7] Alt then proved that these seven postulates hold if and only if: (i) there exists a real-valued function U over combinations of goods such that combination x is preferred to combination y if and only if $U(x)$ is larger than $U(y)$; (ii) the transition from y to x is preferred to the transition from w to z if and only if $U(x)-U(y)$ is larger than $U(z)-U(w)$; (iii) if U* is another real-valued function over combinations of goods that satisfies conditions (i) and (ii), then U* is a linearly increasing transformation of U, that is, $U^*=\alpha U+\beta$. Alt thus provided a mathematically rigorous answer to the question concerning the exact conditions that make utility measurable in the sense envisaged by Lange.

Alt also addressed the validity and empirical verifiability of the seven postulates. He believed that postulate 1 "can be verified *by economic observations*" (431) and is therefore well-founded. In opposition to Bernardelli, Alt found postulate 2 more problematic because it is not clear "whether it is at all possible to make comparisons between transitions […] on the basis of experience" (431). With respect to the other five postulates, Alt left the issue concerning their validity open, arguing that they "can (and must) be tested against experience" (431).

Alt's contribution was distinctive not only in its axiomatic approach, but also in its very conception of measurement. Like Allen (1935), who is not cited in the article, Alt claimed that if the utility numbers are arbitrary only with respect to unit and zero point of measurement, as it is the case when the seven postulates hold, then utility is properly measurable. But, in contrast to Allen, Alt's claim was rooted in an explicit non-classical understanding of measurement according to

[7] Postulates 3 and 6 require that the preference orders over combinations and transitions are transitive and continuous. Postulate 4 connects the two orders by stating that the individual prefers combination x to combination y if and only if he prefers the transition to x to the transition to y whatever the starting combination z is and, at the same time, he prefers reaching whatever combination w by starting from y rather than by starting from x. Postulate 5 provides the preference order over transitions with an additive structure: if transition from x to y is preferred to transition from x' to y', and transition from y to z is preferred to transition from y' to z', then transition from x to z is preferred to transition from x' to z'. Postulate 7 is an Archimedean requirement that, like postulate 4, connects the two preference orders: if x is preferred to y, there exists a finite sequence of equivalent transitions to more preferred combinations such that the last element of the sequence is at least as preferred as x.

which measuring consists of representing numerically the properties of objects rather than assessing ratios:

> When we say that the utility of a commodity is 'measurable' or 'numerically representable' we mean that we can assign a real number […] to each set of commodities […] in such a way that this assignment is unique for choice of origin and unit of measurement. Such an assignment of real numbers is called a function; hence to measure utility is to define a function for all sets of commodities. (Alt [1936] 1971: 424-425)

Similar non-classical conceptions of measurement had previously been suggested by the English philosopher Bertrand Russell (1903), the English physicist Norman Robert Campbell (1928), and the American philosopher of science Ernest Nagel (1931) (Michell 1999). However, in Alt's writings of the 1930s I was not able to find any reference to these authors or their measurement theories. Alt's non-classical conception of measurement appears related rather to the topological theories that Menger and his circle were elaborating in the 1930s. In these theories the properties of a mathematical object, such as the curvature of a curve, were associated with a number that did not change when the object was subjected to certain transformations (see e.g. Menger [1930] 1998 and Alt [1932] 1998).

9.3. Alt's neglect

Even though it provided a solution, at least at the mathematical level, to the utility-measurement issues debated since the 1920s, Alt's contribution was almost completely neglected in subsequent discussion.[8] We can only speculate why this was the case. That the article was published in German in an Austrian journal probably hindered its appreciation in the Anglo-Saxon academic world. Nevertheless, in the early 1930s the *Zeitschrift für Nationalökonomie* did enjoy an international standing and many of our protagonists published in it (e.g. Lange 1932, Allen 1934, Phelps Brown 1936). Furthermore, a one-sentence English abstract of Alt's article appeared in the section "Recent periodicals and new books" of the *Economic Journal* (Anonymous 1936: 574).

The axiomatic style, which was unfamiliar to the economists of the period, may also have fostered neglect. But Alt did make an effort to situate his contribution within the ongoing debate, explaining the meaning and importance of his postulates, and thus making his contribution comprehensible even to mathematically unsophisticated economists.

Finally, Alt did not connect the utility index that is unique up to linear transformations to the expression "cardinal utility". One may speculate that,

[8] In the ten years after its publication, Alt's article was cited only twice in the journals collected in the JSTOR database, i.e. by Stigler 1938 and Tintner 1942.

even if Alt was aware of the cardinal-ordinal terminology, as a mathematician he associated the terms cardinal and ordinal with their mathematical meaning (see above Section 4) and thus did not want to use them in a different sense. In fact, it was the economist Paul Samuelson who, in 1938, coupled "cardinal utility" with utility unique up to linear transformations.

10. Cardinal utility and linear transformations in Samuelson's early work

10.1. Discounting and measuring utility

Samuelson (1915-2009) entered the University of Chicago in 1932 and then moved on to Harvard Graduate School, where he studied under, among others, Joseph Schumpeter and the physicist and mathematical economist Edwin Bidwell Wilson. In 1937 and aged only twenty-one, Samuelson published in the *Review* his first scientific article, "A Note on Measurement of Utility".[9] Here, Samuelson put forward a model of intertemporal choice where the individual behaves so as to maximize the discounted sum of all future utilities. This discounted-utility model quickly became the standard neoclassical formalization of intertemporal choice. Our concern, however, is with the parts of Samuelson's 1937 article that deal with utility measurement.

The connection between utility discounting and utility measurement lies in the circumstance that for Samuelson the maximization of the discounted sum of future utilities implies that the individual is able to rank utility differences, i.e., Pareto's postulate 2:

> Reflection as to the meaning of our Assumption Two [that the individual maximizes the sum of future utilities] will reveal that the individual must make preferences in the Utility dimension itself, that is to say, we must invoke Pareto's Postulate Two, which relates to the possibility of ordering *differences* in utility by the individual. (Samuelson 1937: 160-161)

The above quotation also shows that, following Lange and ignoring the contributions of Phelps Brown and Alt, Samuelson in 1937 identified postulate 2 with the possibility of ranking utility differences. Accordingly, he claimed that postulate 2 restricts the admissible transformations of the utility function to the linearly increasing ones. In particular, Samuelson (1937: 160) contrasted the mere ordinal ranking of utility, which does not provide a measure of utility, with utility invariant to linear transformations, which he considered as properly

[9] In the June 1937 issue of the *Review*, the Danish economist Frederik Zeuthen published a note that, at least in its title, also related to the determinateness-of-the-utility-function debate (Zeuthen 1937). In this note, Zeuthen criticized the behaviorist approach to demand analysis and argued that introspection is necessary to economic theory. However, Zeuthen did not elaborate on issues concerning cardinal utility.

"measurable" (156, 161). However, in his first publication Samuelson did not use the cardinal-ordinal terminology.

10.2. Utility and choices

As is well-known, the 1937 article was the first of an exceptionally copious and long-lasting series. In 1938 alone, Samuelson published four articles in major economics journals, three of which were related to utility theory and demand analysis, while the fourth addressed welfare economics.

The first 1938 article is Samuelson's celebrated "Note on the Pure Theory of Consumer's Behaviour", which appeared in the February 1938 issue of *Economica*. Samuelson (1938a) criticized Hicks's and Allen's demand analysis for not being properly behaviorist, and put forward his own brand of behaviorism, later called the revealed preference approach to consumer demand.[10]

Since the goal of the "Note" was to show that demand analysis requires no reference to utility, Samuelson did not dwell on the issue of utility measurement. However, and this is important for our story, he employed the expression "cardinal utility" for the first time in print. In reviewing the history of demand analysis based on utility, Samuelson (61) argued that it had progressively ruled out unnecessarily restrictive conditions such as "the assumption of the measurability of utility in a cardinal sense". It is not clear, however, what "measurability of utility in a cardinal sense" means, and the expression is not associated with utility invariant to linear transformations.

The second 1938 article, "The Empirical Implications of Utility Analysis", appeared in the October issue of *Econometrica* (Samuelson 1938b). Samuelson argued that the ordinal utility theory initiated by Pareto does have refutable implications in terms of demand behavior, such as the negativity of the substitution effect, and attempted to provide a complete list of these implications. However, Samuelson claimed, the same implications can be derived more easily and directly from the postulates on choices he had put forward in the "Note". In this article Samuelson twice employed the expression "ordinal preference" (345), but not the term "cardinal utility".

10.3. Connecting cardinal utility and linear transformations

Samuelson's third article of 1938, "The Numerical Representation of Ordered Classifications and the Concept of Utility", appeared in the October 1938 issue of the *Review* (Samuelson 1938c) and is particularly relevant for our story because he used the cardinal-ordinal terminology extensively ("cardinal" ten times and

[10] The analysis of Hicks and Allen relied on the assumption that the marginal rate of substitution is decreasing, i.e., that the indifference curves are convex. However, for Samuelson (1938a: 61) this assumption depends on introspection and therefore is not sound; see Moscati 2007.

"ordinal" seven) and consistently coupled the expression "cardinal utility" with utility which is unique up to linearly increasing transformations.

Samuelson sent a draft of the paper to Lange (presumably in early 1938), who replied in the letter of 10 May 1938, mentioned above (Samuelson papers, Box 48). Lange declared Samuelson's manuscript "a contribution which really helps to clarify the subject", acknowledged that Phelps Brown's objections to his 1934 paper were correct, and argued that Postulate 2 was neither necessary nor sufficient for welfare analysis.

At the outset of the article, Samuelson (1938c: 65) acknowledged that Phelps Brown was right in criticizing Lange's results because they were based on an unwarranted identification of the G-numbers representing the ranking of transitions with the difference between the U-numbers representing the ranking of combinations (see Section 7). Now, Samuelson saw that this identification cannot be taken for granted and accordingly investigated under what conditions it is valid. Finally, he discussed the plausibility of these conditions.

In effect, the issue concerning the hypotheses that warrant the identification of the G-numbers with the difference between the U-numbers is exactly the problem that Alt had already addressed and solved in his 1936 article. However, Samuelson did not mention Alt's article.

Samuelson's treatment of the issue was much more informal than Alt's. In Samuelson's paper there is no proper list of postulates concerning the properties of the preference orders over transitions and combinations. More or less implicitly, Samuelson assumed that both orders are complete and transitive. His key postulate endowed the G-numbers with an additive structure. In numerical terms, Samuelson required that if $G(x, y)$ is the number associated with the transition from x to y, and $G(y, z)$ is the number associated with the transition from y to z, then the number $G(x, z)$ associated with the transition from x to z, must be equal to the sum of $G(x, y)$ and $G(y, z)$, i.e., $G(x, y)+G(y, z)=G(x, z)$.

Samuelson (68-70) then showed that this hypothesis is a necessary and sufficient condition to make the G-numbers associated with transitions equal to the difference between the U-numbers associated with combinations, that is, to have $G(x, y)=U(y)-U(x)$. In turn, as Lange had already showed, $G(x, y)=U(y)-U(x)$ if and only if the utility function U is unique only up to linearly increasing transformations.

In the final part of his paper, Samuelson (70) discussed the plausibility of the condition $G(x, y)+G(y, z)=G(x, z)$, and argued that it is an "arbitrary restriction" that must be regarded as "infinitely improbable". Since that condition is necessary and sufficient to make the utility function unique only up to linear transformations, concluded Samuelson, the measurability of utility in the linear

sense envisaged by Lange should also be considered as arbitrary and infinitely improbable.

Samuelson's article contains a terminological novelty that is central to our story: for the first time utility unique up to linearly increasing transformations was explicitly and consistently coupled with the terms "cardinal" and "cardinal measurability". This association occurs ten times in Samuelson's paper, of which two are selected as examples:[11]

> There is […] no need for any cardinal measure of utility. (65)
>
> Dr. Lange has not proved satisfactorily that from these two assumptions [Pareto's postulates 1 and 2] can be derived the cardinal measurability of utility. (66)

We argue, therefore, that "cardinal utility" acquired its current technical meaning in Samuelson's 1938 article.

Three discussions concerning Samuelson's four articles are in order. First, did Samuelson know of Alt's 1936 article? At the least, Samuelson was aware of its existence. In his letter of 10 May 1938, Lange explicitly invited Samuelson to look at Alt's article:

> I would suggest that you look up the article of Alt, Über die Messbarkeit des Nutzens, Zeitschr. F. Nat.- Oeconomie, Bd. VII (1936). (Samuelson's papers, Box 48)

We know from a letter of Ursula Webb Hicks to Samuelson that he did not see the proofs of his article (letter of 4 October 1938, Samuelson's papers, Box 37). Therefore, even if Samuelson looked at Alt's article between May and October 1938, he could not add a reference to Alt. It is noteworthy, however, that in his subsequent writings of the 1930s and 1940s Samuelson did not refer to Alt's 1936 article.[12]

In his *History of economic analysis*, Schumpeter touched upon Samuelson's possible acquaintance with Alt's article. As we have seen (Section 7), Schumpeter appreciated Alt's article; but he was also one of Samuelson's mentors at Harvard and followed his pupil's research on demand analysis. So, it does not seem implausible that, at some point, Schumpeter mentioned Alt's paper to Samuelson. At any rate, in his *History*, after having pointed to Alt's 1936 article as providing a satisfactory solution to the issue of utility

[11] The other eight occurrences can be found at pages 65, 68 and 70 of Samuelson 1938c.

[12] Apparently, Samuelson's first reference to Alt's article is contained in a footnote of Samuelson 1950.

measurement, Schumpeter (1954: 1063) turned to Samuelson's (1938c) paper and declared that "Alt's argument […] was not known to Samuelson".

Secondly, Samuelson's confidence in the plausibility of cardinal utility passed from the agnosticism of his discounted-utility article of February 1937 (where, in effect, cardinal utility was necessary to make sense of the discounted-utility model) to the disbelief expressed in the *Review* article of October 1938. Despite shifting opinion on the plausibility of cardinal utility, Samuelson's views on its measurability did not change, and he consistently maintained that utility invariant to linear transformations (if existing) is properly measurable.

Therefore, we can affirm that from at least 1937 Samuelson had overcome the classical understanding of measurement, although he did not provide an explicit new definition of measurement. As mentioned in Section 9.2, a non-classical conception of measurement had been suggested by physicist Norman Campbell in a series of books published in the 1920s. Thus, one may speculate that Samuelson's non-classical understanding of measurement may be related to the knowledge of physics he had acquired during his undergraduate studies at Chicago, or through discussion with his physicist mentor, Wilson. However, in Samuelson's papers I have found no archival evidence supporting this hypothesis.

Thirdly, Samuelson's pivotal role in defining the current meaning of cardinal utility and, more generally, his intensive dealing with utility analysis in the late 1930s, show the inadequacy of the conventional account, according to which Samuelson's chief goal in this period was to free economic theory from any vestige of the utility concept (see e.g. Wong 2006). This may have been the goal of the 1938 "Note", which however constituted only a part of the larger and quite eclectic research agenda that Samuelson was pursuing in his early twenties. The fact that the young Samuelson was fully engaged with utility analysis is confirmed by the central role that he took in the early 1940s in stabilizing the new meaning of cardinal utility.

11. Cardinal utility between 1938 and 1944

This Section briefly reviews the last sparks of the determinateness-of-the-utility-function debate, and surveys the references to cardinal utility between 1938 and 1944.

In an article published in *Economica* in May 1938, Bernardelli reiterated his 1934 defense of marginal-utility concepts against Pareto's ordinalism and the behaviorism of Hicks and Allen. Now, however, Bernardelli (1938: 209) discarded Postulate 2 as "open to serious objection", and claimed that marginal-utility concepts can be saved by positing three allegedly less objectionable axioms

regarding marginal utility itself. Again, Bernardelli did not employ the expression "cardinal utility". Samuelson (1939) promptly criticized Bernardelli's approach in the February 1939 issue of *Economica*, arguing that Bernardelli's postulates are open to precisely the same objections as Postulate 2, and that therefore "the new definitions leave matters just where they were." (87) Notably, to characterize the total utility function implied by Bernardelli's postulates, Samuelson (86) used the expression "cardinal index of utility".

In 1938, Allen published *Mathematical analysis for economists*, a handbook in mathematics aimed at economists. Allen (1938: 3-14) used the term "cardinal", but in reference to the mathematical notion of number, which, in turn, was associated with counting objects and their measurement in the classical sense. More generally, Allen left no room for cardinal utility; utility was presented as "ordinal and non-measurable" (126) and conceived of as deducible in principle from the marginal rate of substitution, which remained the fundamental concept of Allen's behaviorist analysis.

In 1939 Hicks published *Value and capital*, which brought to a close the Ordinal Revolution initiated by Pareto. Here Hicks (1939: 11-52) abandoned the behaviorist approach of his 1934 joint article with Allen, and provided the formulation of demand analysis in terms of ordinal utility that quickly became canonical in economics. Not even in *Value and capital* is there any room for cardinal utility.

In 1939, the Cambridge social anthropologist and economist Wallace E. Armstrong published "The determinateness of the utility function" in the *Economic Journal*, the last article reiterating the title of Lange's 1934 paper. Similarly to Bernardelli in 1934, Armstrong defended introspection as a scientifically legitimate source of knowledge that allowed him to observe "that I prefer *A* to *B* more strongly than I prefer *A* to *C*", which was "a sufficient reason for the assumption of a determinate utility function" (1939: 462), i.e., of a utility function invariant to linear transformations. However, Armstrong did not label such a utility function "cardinal".

In 1940 Frank Knight, one of the leaders of the Chicago School, entered the debate initiated by Lange (who was then also at Chicago). In an article in the *Journal of Political Economy*, Knight (1940) attacked the application of the positivistic methods of natural science to economics and defended the role of introspection in understanding economic phenomena. In making this point, Knight mentioned "the distinction between cardinal and ordinal magnitudes", and argued that utility "is not measurable, in the sense in which any physical magnitude is measured." (19) Although Knight referred in a footnote to Lange's 1934 article and the discussion it generated (19), he left implicit what he meant by "cardinal". In "Realism and relevance in the theory of demand", published in

1944 in the same journal, Knight was more explicit. He criticized the ordinal approach to demand analysis epitomized by Hicks's *Value and capital* and, referring appreciatively to Bernardelli's 1934 article, he advocated the psychological realism of decreasing marginal utility as well as its relevance for understanding demand behavior. In a number of passages, Knight (1944: 290, 301-302, 304) associated decreasing marginal utility with the ranking of utility differences and the conception of utility as a "cardinal magnitude".

In November 1940, after he had already left Harvard for the Massachusetts Institute of Technology, Samuelson delivered his Harvard Ph.D. dissertation, "Foundations of analytical economics. The observational significance of economic theory" (Samuelson 1940), which, seven years later, became his *magnum opus*: *Foundations of economic analysis* (Samuelson 1947). In the dissertation Samuelson played down the revealed-preference approach proposed in his 1938 "Note", and presented the theory of consumer's demand more or less according to the ordinal-utility approach. Accordingly, after having shown that "the content of the utility analysis in its most general form [involves] only an ordinal preference field" (1940: 147), Samuelson examined some "special and extra assumptions" (147) introduced into the analysis by a number of writers. The first was cardinal utility, explicitly defined as utility "unique except for constants of scale and origin" (149). Although he maintained a skeptical stance on the empirical validity and theoretical usefulness of cardinal utility, Samuelson investigated in detail its connection with other special assumptions of utility theory, such as the independence of the marginal utilities of goods and the constancy of the marginal utility of income. In particular, Samuelson showed that the independency of marginal utilities restricts the arbitrariness of the utility indices to linear transformations of the form $F(U)=\alpha U+\beta$. Samuelson's 1940 discussion of cardinal utility passed with almost no modification into the *Foundations* (Samuelson 1947: 173-202).

In 1942, Chicago University Press published a volume in memory of Schultz, who had died in a car accident in 1938. The volume collected papers by Lange, Allen, Samuelson, Milton Friedman, Jacob Marschak, Jan Tinbergen and other distinguished economists of the period, many of whom were associated with the Cowles Commission. In his contribution, Samuelson (1942) critically analyzed the notion of constancy of the marginal utility of income along the lines already developed in his 1941 dissertation and, in the course of the analysis, referred three times to "the cardinal index of utility" (75-76).

In 1943, Robert L. Bishop, a young colleague of Samuelson at MIT, published "Consumer's surplus and cardinal utility" in the *Quarterly Journal of Economics*, the first economics article containing the expression cardinal utility in its title. Bishop (1943) argued that the cardinal measurability of utility is a necessary condition to make sense of the notion of consumer's surplus, which he

considered useful for welfare analysis. Bishop also claimed that cardinal utility draws not only from the implausible assumption that the marginal utilities of all commodities are independent, but also from more acceptable hypotheses that posit "a limited amount of independence" (438) among marginal utilities.

This brief survey of the economics literature between 1938 and 1944 has shown that, in this period, there were numerous and significant references to cardinal utility, and that cardinal utility was usually intended in the specific sense established by Samuelson (1938c), i.e., as utility unique up to linear transformations. Moreover, although he remained critical of cardinal utility, Samuelson consistently referred to it as utility unique up to linear transformations, and thus contributed to the stabilization of the current meaning of cardinal utility.

To be sure, cardinal utility remained peripheral in the 1938-1944 economics literature. In fact, it was at odds with the ordinal approach that, especially after the publication of Hicks's *Value and capital*, dominated demand analysis. The voices of critics of ordinalism and supporters of cardinal utility, such as Bernardelli, Armstrong, or Knight, were not very influential. Things changed when, in *Theory of games and economic behavior* (1944), John von Neumann and Morgenstern put forward their Expected Utility Theory. This theory showed that cardinal utility draws not only from the assumption that individuals are capable of ranking utility differences between riskless alternatives or that the marginal utilities of commodities are independent, but also from a set of apparently less problematic axioms concerning preferences over risky alternatives. The rise of Expected Utility Theory propelled the use of cardinal utility and began a new phase in the history of utility analysis. These developments, however, must be studied in another paper.

12. Conclusions

This paper has shown that cardinal utility entered economic analysis during the Ordinal Revolution initiated by Pareto and not, as many popular histories of utility theory assume, before it. In so doing, the paper has contributed to the history of both cardinal utility and the Ordinal Revolution. In particular, the paper has illustrated how, in the 1920s, 1930s and early 1940s, the opponents of strictly ordinal and behaviorist approaches to demand analysis advocated cardinal utility as a mean to preserve some concepts of earlier marginal utility theory without returning to the idea that utility is measurable in the classical sense.

The paper has also demonstrated that Samuelson had a pivotal role, not only in defining the current meaning of cardinal utility but also in stabilizing that meaning. As far as I know, the present study is the first to call attention to the

connection between Samuelson and cardinal utility. In addition, the paper has challenged a widespread image according to which Samuelson's chief goal during the Ordinal Revolution was to free economics from any vestige of the utility concept.

Furthermore, the paper has brought to light the figure of Alt, whose significant contribution to utility analysis and interesting position at the crossroads of mathematics and economics have been somehow neglected by existing histories of utility theory. The paper has also investigated the relationships between Alt's and Samuelson's contributions to the definition of cardinal utility, and provided archival evidence that in 1938 Samuelson was made aware of the existence of Alt's 1936 article.

Finally, the paper has shown that in some of the articles that led to the definition of cardinal utility, the first clear signs of a non-classical understanding of measurement appeared in utility theory. However, it was only after 1944, during a long discussion of the nature of the utility function featuring in von Neumann's and Morgenstern's Expected Utility Theory, that utility theorists consciously and definitely abandoned the classical understanding of measurement. These post-1944 developments in the history of utility measurement will be investigated in another paper.

References

Allen, R.G.D. 1932. The Foundations of a Mathematical Theory of Exchange. *Economica*, 36: 197–226.

Allen, R.G.D. 1934. Nachfragefunktionen für Güter mit korreliertem Nutzen. *Zeitschrift für Nationalökonomie*, 5: 486–506.

Allen, R.G.D. 1935. A Note on the Determinateness of the Utility Function. *Review of Economic Studies*, 2: 155–158.

Allen, R.G.D. 1938. *Mathematical Analysis for Economists*. London: Macmillan.

Alt, F. [1932] 1998. Zur Theorie der Krummung. *Ergebnisse eines Mathematischen Kolloquiums*, 1931/1932. In Menger 1998: 186.

Alt, F. 1935. Industrielle Standortsfragen. *Zeitschrift für Nationalökonomie*, 6: 85–97.

Alt, F. [1936] 1971. On the Measurability of Utility. In Chipman and others 1971: 424–431.

Alt, F., and A. Akera. 2006. Franz Alt interview: January 23 and February 2, 2006. *ACM Oral History interviews*, 1: 1–89.

Amoroso, L. 1921. *Lezioni di economia matematica*. Bologna: Zanichelli.

Anonymous, 1936. Recent Periodicals and New Books. *Economic Journal*, 46: 568–587.

Armstrong, W.E. 1939. The Determinateness of the Utility Function. *Economic Journal*, 49: 453–467.

Benham, F.C. 1934. Notes on the Pure Theory of Public Finance. *Economica*, NS, 1: 436–458.

Bernardelli, H. 1934. Notes on the Determinateness of the Utility Function, II. *Review of Economic Studies*, 2: 69–75.

Bernardelli, H. 1938. The End of the Marginal Utility Theory? *Economica*, NS, 5: 192–212.

Bishop, R.L. 1943. Consumer's Surplus and Cardinal Utility. *Quarterly Journal of Economics*, 57: 421–449.

Bowley, A.L. 1924. *The Mathematical Groundwork of Economics*. Oxford: Claredon Press.

Bruni, L., and F. Guala. 2001. Vilfredo Pareto and the Epistemological Foundations of Choice Theory. *History of Political Economy*, 33: 21–49.

Campbell, N.R. 1928. *An Account of the Principles of Measurement and Calculation*. London: Longman, Green and Co.

Cantor, G. [1887] 1932. Mitteilungen zur Lehre vom Transfiniten. In G. Cantor, *Gesammelte Abhandlungen*, edited by E. Zermelo, 378–439. Berlin: Springer.

Chipman, J.S. and others, eds. 1971. *Preference, Utility and Demand*. New York: Harcourt Brace Jovanovich.

de Pietri-Tonelli, A. 1927. *Traité d'économie rationnelle*. Paris: Giard.

Dobb, M.H. 1966. Oskar Lange. *Journal of the Royal Statistical Society*. Series A, 129: 616–617.

Donoghue, M. 2007. Harro Bernardelli (1906–1981). In *A Biographical Dictionary of Australian and New Zealand Economists*, edited by J.E. King, 25–28. Cheltenham: Elgar.

Edgeworth, F.Y. 1894. Professor J.S. Nicholson on 'Consumers' Rent', *Economic Journal*, 4: 151–158.

Edgeworth, F.Y. 1900. The Incidence of Urban Rates, I. *Economic Journal*, 10: 172–193.

Edgeworth, F.Y. 1907. Appreciations of Mathematical Theories. *Economic Journal*, 17: 221–231.

Edgeworth, F.Y. 1915. Recent Contributions to Mathematical Economics, I. *Economic Journal*, 25: 36–63.

Edgeworth, F.Y. 1925. *Papers Relating to Political Economy*, 3 vols. London: Macmillan.

Fishburn, P.C. 1970. *Utility Theory for Decision Making*. New York: Wiley.

Fisher, I. 1892. *Mathematical Investigations in the Theory of Value and Prices*. New Haven: Yale University Press.

Fisher, I. 1927. A Statistical Method for Measuring 'Marginal Utility' and Testing the Justice of a Progressive Income Tax." In *Economic Essays Contributed in Honor of John Bates Clark*, edited by J. Hollander, 157–193. New York: Macmillan.

Frisch, R. [1926] 1971. On a Problem in Pure Economics. In Chipman and others 1971:

386–423.
Frisch, R. 1932. *New Methods of Measuring Marginal Utility*. Tübingen: Mohr.
Giocoli, N. 2003. *Modeling Rational Agents*. Cheltenham: Elgar.
Hancock, K., and J.E. Isaac. 1998. Sir Henry Phelps Brown, 1906–1994. *Economic Journal*, 108: 757–778.
Hands, W. 2010. Economics, Psychology, and the History of Consumer Choice Theory. *Cambridge Journal of Economics*, 34: 633–648.
Hands, W. 2011. Back to the Ordinalist Revolution. *Metroeconomica*, 62: 386–410.
Hayek, F.A. von. 1934. Carl Menger. *Economica*, NS, 1: 393–420.
Hicks, J.R. 1939. *Value and Capital*. Oxford: Clarendon Press.
Hicks, J.R. 1981. Prefatory note. In J.R. Hicks, *Wealth and* Welfare, 3–5. Oxford: Blackwell.
Hicks, J.R., and R.G.D. Allen. 1934. A Reconsideration of the Theory of Value. *Economica*, NS, 1: 52–76, 196–219.
Howson, S. 2011. *Lionel Robbins*. Cambridge: Cambridge University Press.
Ingrao, B., and G. Israel. 1990. *The Invisible Hand*. Cambridge, MA: MIT Press.
Knight, F.H. 1940. "What is Truth" in Economics? *Journal of Political Economy*, 48: 1–32.
Knight, F.H. 1944. Realism and Relevance in the Theory of Demand. *Journal of Political Economy*, 52: 289–318.
Lange, O. 1932. Die allgemeine Interdependenz der Wirtschaftsgrössen und die Isolierungsmethode. *Zeitschrift für Nationalökonomie*, 4: 52–78.
Lange, O. 1934a. The Determinateness of the Utility Function. *Review of Economic Studies*, 1: 218–225.
Lange, O. 1934b. Notes on the Determinateness of the Utility Function, III. *Review of Economic Studies*, 2: 75–77.
Lenfant, J.-S. 2006. Complementarity and Demand Theory: From the 1920s to the 1940s. *History of Political Economy*, 38 (Supplement): 48–85.
Lenfant, J.-S. 2012. Indifference Curves and the Ordinalist Revolution. *History of Political Economy*, 44: 113–155.
Leonard, R. 2010. *Von Neumann, Morgenstern, and the Creation of Game Theory*. New York: Cambridge University Press.
Lutfalla, G. 1933. Compte rendu de la deuxième réunion européenne de l'Econometric Society. *Revue d'économie politique*, 47: 173–192.
Mandler, M. 1999. *Dilemmas in Economic Theory*. New York: Oxford University Press.
Marschak, J. 1934. The Meetings of the Econometric Society in Leyden, September–October, 1933. *Econometrica*, 2: 187–203.
Menger, K. [1930] 1998. Entwurf einer neuen Theorie des Maßes. *Ergebnisse eines Mathematischen Kolloquiums*, 1929/1930. In Menger 1998: 124–128.
Menger, K. 1998. *Ergebnisse eines Mathematischen Kolloquiums*, edited by E. Dierker and K. Sigmund. Wien–New York: Springer.
Michell, J. 1999. *Measurement in Psychology*. Cambridge: Cambridge University Press.
Montesano, A. 2006. The Paretian Theory of Ophelimity in Closed and Open Cycles. *History of Economic Ideas*, 14: 77–100.
Morgenstern, O. 1931. Die drei Grundtypen der Theorie des subjektiven Wertes. *Schriften des Vereins für Sozialpolitik*, 183: 1–42.
Moscati, I. 2007. History of Consumer Demand Theory 1871–1971: A Neo-Kantian Rational Reconstruction. *European Journal of the History of Economic Thought*, 14: 119–156.
Moscati, I. 2013. Were Jevons, Menger and Walras really Cardinalists? *History of Political Economy*, forthcoming.
Nagel, E. 1931. Measurement. *Erkenntnis*, 2: 313–333.
Neumann, J. von, and O. Morgenstern. 1944. *Theory of Games and Economic Behavior*. Princeton: Princeton University Press.
Niehans, J. 1990. *A History of Economic Theory*. Baltimore, MD.: Johns Hopkins University Press.
Pareto, V. [1892] 2007. *Considerations on the Fundamental Principles of Pure Political Economy*. New York: Routledge.

Pareto, V. [1896] 1964. *Cours d'économie politique*. Tome I. Geneve: Droz.
Pareto, V. [1900] 2008. Summary of Some Chapters of a New Treatise on Pure Economics by Professor Pareto. *Giornale degli Economisti*, 67: 453–504.
Pareto, V. [1909] 1971. *Manual of Political Economy*. New York: Kelley.
Phelps Brown, E.H. 1934a. Demand Functions and Utility Functions. *Econometrica*, 2: 51–58.
Phelps Brown, E.H. 1934b. Notes on the Determinateness of the Utility Function, I. *Review of Economic Studies*, 2: 66–69.
Phelps Brown, E.H. 1936. Review of J.W.F. Rowe, *Markets and Men,* Cambridge University Press. *Zeitschrift für Nationalökonomie*, 7: 695–697.
Rosenstein-Rodan, P. [1927] 1960. Marginal Utility. *International Economic Papers*, 10: 71–106.
Russell, B. 1903. *The Principles of Mathematics*. Cambridge: Cambridge University Press.
Samuelson, P.A. 1937. A Note on Measurement of Utility. *Review of Economic Studies*, 4: 155–161.
Samuelson, P.A. 1938a. A Note on the Pure Theory of Consumer's Behaviour. *Economica*, NS, 5: 61–71.
Samuelson, P.A. 1938b. The Empirical Implications of Utility Analysis. *Econometrica*, 6: 344–356.
Samuelson, P.A. 1938c. The Numerical Representation of Ordered Classifications and the Concept of Utility. *Review of Economic Studies*, 6: 65–70.
Samuelson, P.A. 1939. The End of Marginal Utility: A Note on Dr. Bernardelli's Article. *Economica*, NS, 6: 86–87.
Samuelson, P.A. 1940. Foundations of Analytical Economics. The Observational Significance of Economic Theory (Ph.D. dissertation). Samuelson Papers, Box 91.
Samuelson, P.A. 1942. Constancy of the Marginal Utility of Income. In *Studies in Mathematical Economics and Econometrics*, edited by O. Lange, F. McIntyre, and Th.O. Yntema, 75–91. Chicago: Chicago University Press.
Samuelson, P.A. 1947. *Foundations of Economic Analysis*. Cambridge, MA: Harvard University Press.
Samuelson. P.A. 1950. Probability and the Attempts to Measure Utility. *Economic Review*, 1: 167–173.
Samuelson. P.A. Papers, 1933–2010 and undated. David M. Rubenstein Rare Book & Manuscript Library, Duke University, Durham, North Carolina.
Schmidt, T., and C.E. Weber. 2008. On the Origins of Ordinal Utility: Andreas Heinrich Voigt and the Mathematicians. *History of Political Economy*, 40: 481–510.
Schmidt, T., and C.E. Weber. 2012. Andreas Heinrich Voigt and the Hicks–Allen Revolution in Consumer Theory. *Economic Inquiry*, 50: 625–640.
Schröder, E. 1873. *Lehrbuch der Arithmetik und Algebra.* Leipzig: Teubner.
Schumpeter, J.A. 1954. *History of Economic Analysis*. New York: Oxford University Press.
Stigler, G.J. 1938. Social Welfare and Differential Prices. *Journal of Farm Economics*, 20: 573–586.
Tintner, G. 1942. A Contribution to the Non-Static Theory of Choice. *Quarterly Journal of Economics*, 56: 274–306.
Voigt, A. [1893] 2008. Number and Measure in Economics. In Schmidt and Weber 2008.
Wong, S. 2006. *The Foundations of Paul Samuelson's Revealed Preference Theory*. New York: Routledge.
Zeuthen, F. 1937. On the Determinateness of the Utility Function. *Review of Economic Studies*, 4: 236–239.