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Applying default probabilities in an exponential barrier structural model

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Abstract
This paper shows that the use of a time-dependant barrier in a structural model improve its flexibility because it allows to incorporate, as input, the probability of default. The main result achieved is the assessment that the default barrier is, indeed, characterized by a non flat structure.

JEL Classification: G13, G33

Keywords: default structural models, barrier options with exponential boundaries, implied default probability

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1) Introduction and motivation

The evaluation of defaultable debt and the prediction of bankruptcy have become, in the last years, an extremely important topic in financial literature. Quantitative methodologies developed to analyze this problem can be divided in two groups: structural and reduced models. In the first case, default is assumed to be an endogenous event occurring when, loosely speaking, it is realized that the value of the assets of a company cannot fully cover its outstanding debt. The second group encompasses models in which default is triggered by an exogenous random factor such as, for instance, the first jump of a Poisson jump process. This article relies on the first approach and aims at extending and improving some aspects of known results in financial literature.

Structural models basic assumption is that the value of the company's assets evolves according to a stochastic process that can be observed only at specific points in time, and that equity and debt can be represented as derivative contracts written on the assets. Exploiting some standard valuation techniques it is possible to provide expressions for the value of the equity, the outstanding debt and the default probability. The first attempt to apply option pricing methodology, developed by Black and Scholes (1973) and Merton (1973), for the valuation of default-prone debt issued by a firm has been presented by Merton (1974). In this work the value of the equity of a company is seen as the price of a European call option. Such value is positive if assets are large enough to pay back in full the company’s debt; otherwise assets are used in full to repay part of the debt, leaving no extra amount to be given to stockholder. The ‘plain vanilla’ nature of the option implies that default can occur only when the option expires. To circumvent this limitation, Black and Cox (1976) presented a more sophisticated model in which
the presence of a time-varying exponential barrier is considered. If, at any time before maturity, the value of the assets crosses such barrier, the company defaults. The value of the equity is, then, the price of a down-and-out barrier option and default before maturity is properly considered.

Barrier option pricing has been widely investigated in financial literature: the first formula, obtained assuming that the value of the underlying asset follows a geometric Brownian motion and the barrier is constant, has been proposed by Merton (1973). Alternative expressions for pricing these options have been obtained, among others, by Reiner and Rubinstein (1991). When the dynamics of the underlying is more complex, including for instance stochastic volatility and jumps, closed-form formulae for the price cannot be obtained; some numerical techniques must then be used instead. For instance, Moretto et al. (2010) develop a multinomial tree valuation methodology for constant barrier options when the underlying asset is characterized by stochastic volatility while D’Ippoliti et al. (2010) exploit the ‘exact algorithm’ proposed by Broadie and Kaya (2006) to price options with constant barrier when the underlying presents stochastic volatility and jumps in both underlying and volatility processes. The tree pricing methodology has been also applied to structural models: Cenci and Gheno (2005) provide a framework to evaluate an amortization in which the debtor could become insolvent before full reimbursement is achieved.

If the strike price of the option involved in finding the value of the equity is simple to determine, being the value at maturity of the outstanding debt, the barrier level is usually assumed to be unknown and its determination is crucial. Brockman and Turtle (2003) adopt the formula for pricing barrier option developed by Merton (1973) to evaluate equities as a down-and-out call, compute default barrier levels for a large
number of U.S. industrial firms on NYSE, AMEX and Nasdaq, and find conclusive empirical evidence that including barriers is both statistically and economically significant. Reisz and Perlich (2007) criticize the results by Brockman and Turtle because the barriers obtained with this approach are “uniformly larger than the amount of debt outstanding”. If default occurs because the value of the assets is not large enough to full repay the debt at maturity, the value of the barrier should resemble the value of the debt and be smaller or equal to its face value. Reisz and Perlich apply instead the KMV methodology (Crosbie and Bohn, 2002) that is a refinement of the approach by Merton (1974) in which the debt structure of the company is transformed into a zero-coupon debt with a given maturity. The authors find that the values of the barrier are significantly different from zero, being on average equal to 30% of the market value of the firms’ assets. Finally, Bharath and Shumway (2008) test if the distance to default has some forecasting capacity in terms of bankruptcy prediction.

Our analysis starts from a different hypothesis: instead of searching for a level of the default barrier, we impose that the value of the barrier at maturity must be equal to the face value of the debt. Following Black and Cox, we apply a non constant barrier; this allows introducing a new variable: the barrier curvature parameter. Such value, along with the unknown values of assets and its volatility, are to be computed. This is achieved using the default probability, derived using the credit default swap written on the company’s debt, as an input.

This article is structured as follows: section 2 presents the model and the related formulae; section 3 reports and discusses the numerical results. Finally section 4 concludes.
2) The model

We exploit a slightly simplified version of the structural model by Black and Cox (1976), where default can occur in two ways: if a stochastic process either crosses a lower time-varying barrier or, at a given maturity, if it ends up being less than a reference value. Consider a company whose assets value evolves according to the geometric Brownian motion

\[ dV(t) = \alpha V(t) dt + \sigma V(t) dW(t) \]  

(1)

where \( \alpha \) and \( \sigma > 0 \) are respectively the drift and volatility parameters and \( W(t) \) is a Wiener process\(^1\). Assets are funded partially through a default-prone debt with face value \( B \), expected to be reimbursed at maturity \( T \) that pays no intermediate flows before expiration, and equities given to the company by stockholders. Let \( D(t,V(t)) \) be the time \( t \leq T \) market value of the debt with face value \( B \) and maturity \( T \) and \( E(t,V(t)) = V(t) - D(t,V(t)) \) the time \( t \) market value of the equity. As in Merton (1974), bankruptcy occurs if, at maturity, the value of the assets is not large enough to fully repay creditors: this happens if, in \( T \), if \( V(T) < B \). The company defaults also if the value of its assets becomes, for the first time, smaller than the value of the exponential barrier \( C(t) = C e^{-\gamma (T-t)} \), being \( V(0) > C(0) \), where \( C \) is the level of the barrier in \( T \) and \( \gamma \) is the barrier curvature parameter. Let \( \tau = \inf \{ t \geq 0 | V(t) \leq C(t); V(0) > C(0) \} \) be the first moment \( V(t) \) hits barrier \( C(t) \) and denote with \( \tau^* \) the moment default occurs.

Three cases can possible: if \( \tau \leq T \) then \( \tau^* = \tau \), if \( \tau > T \) and \( V(T) < B \) then \( \tau^* = T \); finally, if \( \tau > T \) and \( V(T) \geq B \), \( \tau^* = +\infty \).

\(^1\) For sake of simplicity we omit the ‘dividend yield’ parameter.
Both $D(t,V(t))$ and $E(t,V(t))$ can be evaluated using a contingent claim approach; in fact such values can be regarded as derivative contracts written on the company’s assets.

Boundary conditions for the debt are $D(T,V(T)) = \min[V(T); B]$ and, if $\tau^* < T$, $B(\tau^*, V(\tau^*)) = B(\tau^*, C(\tau^*)) = C(\tau^*)$ while for the equity are $E(T,V(T)) = \max[V(T) - B; 0]$ and, again if $\tau^* < T$, $E(\tau^*, C(\tau^*)) = 0$. It is easy to see that the value of the equity is related to a down-and-out call barrier option with a time-varying barrier $C(t)$ and no rebate.

As said above we decide to fix the value of the barrier at maturity equal to the face value of the debt: $C(T) = C = B$. Under dynamics (1), Black and Cox, as well as Bielecki and Rutkowski (2003) and Lando (2004), a closed-form expression for $D(t,V(t))$ and a formula for the probability of default can be obtained. Assuming $V(t)$ has no dividend yield and recalling that $C(T) = C = B$, the value in $t$ of the debt is

$$D(t,V(t)) = B e^{-(r-\gamma)T} \left[ N(z_1(V(t)) - y(t)^{\delta^2} N(z_3(V(t))) + V(t) \{ N(z_3(V(t))) \} \right] + y(t)^{\delta^2} N(z_1(V(t)))] + V(t) \{ N(z_3(V(t))) \} \right]$$

where $y(t) = \frac{C(t)}{V(t)}$, $\theta = \frac{r-\gamma + 0.5\sigma^2}{\sigma^2}$, $\delta = (r-\gamma - 0.5\sigma^2)^2 + 2\sigma^2(r-\gamma)$, $\xi = \sqrt{\frac{\delta}{\sigma^2}}$, $\gamma_1 = \frac{\ln V(t) + (r-0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$, $\gamma_2 = \frac{\ln B e^{-2\gamma(T-t)} + (r-0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$, $\gamma_3 = \frac{\ln B + (r-0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$, $\gamma_4 = \frac{\ln B e^{-2\gamma(T-t)} + (r+0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$, $\gamma_5 = \frac{\ln C e^{-\gamma(T-t)} + \ln + \xi^2 + \gamma_2^2 (T-t)}{\sigma\sqrt{T-t}}$, $r$ is the risk-less rate of return, and
\[ N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-0.5z^2} dz. \] It is straightforward to get, from (2) the expression for the value of the equity

\[
E(t; V(t)) = V(t) - D(t; V(t)) \\
= V(t) \left[ 1 - N(z_v(t)) \right] - \left[ (y(t)^{\delta} - y(t)^{\theta}) N(z_v(t)) \right] \\
- Be^{-\gamma(t-\theta)} \left[ N(z_v(t)) \right] - \left[ (y(t)^{\delta} - y(t)^{\theta}) N(z_v(t)) \right].
\]

The probability that the company has not gone bankruptcy before time \( x \) or, in other words, the probability that \( V(y) \geq C(y) \) for all \( t \leq y \leq x \leq T \) is

\[
\Pr[V(y) \geq C(y)] = N \left\{ \ln \frac{V(t)}{Ce^{-\gamma(t-\theta)}} \left[ \frac{(r - 0.5\sigma^2)(x-t)}{\sigma \sqrt{x-t}} \right] \right\} - \left[ \frac{V(t)}{Ce^{-\gamma(t-\theta)}} \right]^{-2(r-\gamma)\sigma^2} \cdot N \left\{ \ln \frac{Ce^{-\gamma(t-\theta)}}{V(t)e^{-\gamma(t-\theta)}} \left[ \frac{(r - 0.5\sigma^2)(x-t)}{\sigma \sqrt{x-t}} \right] \right\}
\]

So that the probability of default between \( t \) and \( x \) is \( PD(t; x) = 1 - \Pr[V(y) \geq C(y)] \), for all \( y \) such that \( t \leq y \leq x \leq T \).

Value \( V(t) \) and parameter \( \sigma \) in (1), as well as the barrier curvature \( \gamma \), are not directly observable. However, they can be recovered following a standard methodology: assume to know \( E(t) \) and \( \sigma_E \) of the stochastic process followed by the value of the equity

\[
dE(t) = \alpha E(t) dt + \sigma_E dW(t).
\]

Ito’s lemma assures that the Brownian motion in the above dynamics and the one in (1) are the same and that. Further, as \( E(t) \) is a function of \( V(t) \) it is possible to write

\[
dE(t) = \left[ \frac{\partial}{\partial t} E(t) + \frac{\partial}{\partial V(t)} E(t) \cdot \alpha V(t) + \frac{1}{2} \frac{\partial^2}{\partial V^2(t)} E \cdot \sigma V(t) \right] dt \\
+ \frac{\partial}{\partial V(t)} E(t) \cdot \sigma V(t) dW(t).
\]
The volatility terms of $dE(t)$ and $dV(t)$ must be the same so that the following equality

$$\sigma_E E(t) = \Delta_E \sigma V(t),$$

where $\Delta_E = \frac{\partial}{\partial V(t)} E(t)$ is the delta, i.e. the first derivative of the equity with respect to the value of the underlying asset, must hold. Such derivative is

$$\frac{\partial}{\partial V(t)} E = 1 - N(z_3(V(t))) + \frac{n(z_3(V(t)))}{\sigma \sqrt{T-t}} + \left[ y(t) \right]^{\theta} \left[ 2(\theta - 1)N(z_4(V(t))) + \frac{n(z_4(V(t)))}{\sigma \sqrt{T-t}} \right]$$

$$- \left[ y(t) \right]^{\theta+\xi} \left[ (1 - \theta - \xi)N(z_5(V(t))) - \frac{n(z_5(V(t)))}{\sigma \sqrt{T-t}} \right]$$

$$- \left[ y(t) \right]^{\theta-\xi} \left[ (1 - \theta + \xi)N(z_5(V(t))) - \frac{n(z_5(V(t)))}{\sigma \sqrt{T-t}} \right]$$

$$- \frac{Be^{-r(T-t)}}{V(t)} \left[ n(z_3(V(t))) \right] \left( \frac{\sigma}{\sqrt{T-t}} \right) + \left[ y(t) \right]^{\theta-2} \left[ (2\theta - 2)N(z_2(V(t))) - \frac{n(z_2(V(t)))}{\sigma \sqrt{T-t}} \right]$$

where $n(x) = N'(x) = \frac{1}{\sqrt{2\pi}} e^{-0.5x^2}.$

The second equation to be solved is, simply, the one given by (3) the theoretical value of the equity and its observed market capitalization $E_{obs}(t, V(t)).$ Finally, the last equation comes from equating the theoretical default probability as given by the model and the market one, implied in the credit default swap (CDS) quotations, $PD_{obs}(t; T).$

To find $V(t), \sigma$ and $\gamma$ we have to solve the following non-linear system of equations

$$\begin{cases}
E(t, V(t)) = E_{obs}(t, V(t)) \\
\sigma_E E(t, V(t)) = \sigma V(t, V(t)) \Delta_E \\
PD(t; T) = PD_{obs}(t; T)
\end{cases} \quad (4)$$

This will be done in the following section, where an analysis of the results is also performed.

3) **Numerical results**
Our objective in this section is twofold: we first try to solve system (4) and then we discuss the numerical results. For a similar analysis with constant barrier refer to Agosto (2009). The claim is that if an exponential barrier fits appropriately market data, the curvature parameter of the barrier should be different from zero.

3.1) Data

We focus our attention on a panel composed of seven companies (chosen from three different sectors: Industry, Utilities, Automobile) whose stocks are listed in the Euro Stoxx 50 Index. Such companies are Bayer, Daimler, Enel, E.on, Philips, Siemens and Volkswagen.

<table>
<thead>
<tr>
<th>Company</th>
<th>Date</th>
<th>Equity mkt cap.</th>
<th>Debt Value</th>
<th>Impl. vol.</th>
<th>Prob. def.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bayer</td>
<td>03-2009</td>
<td>27 791.57</td>
<td>19 023</td>
<td>0.3745</td>
<td>0.01143</td>
</tr>
<tr>
<td></td>
<td>06-2009</td>
<td>31 605.95</td>
<td>14 139</td>
<td>0.3034</td>
<td>0.0078</td>
</tr>
<tr>
<td>Daimler</td>
<td>03-2009</td>
<td>20 593.34</td>
<td>74 091</td>
<td>0.6348</td>
<td>0.0338</td>
</tr>
<tr>
<td></td>
<td>06-2009</td>
<td>27 288.03</td>
<td>73 696</td>
<td>0.4691</td>
<td>0.02264</td>
</tr>
<tr>
<td>Enel</td>
<td>03-2009</td>
<td>22 348.44</td>
<td>59 672</td>
<td>0.4539</td>
<td>0.04382</td>
</tr>
<tr>
<td></td>
<td>06-2009</td>
<td>32 629.65</td>
<td>66 237</td>
<td>0.3141</td>
<td>0.01615</td>
</tr>
<tr>
<td>E.on</td>
<td>03-2009</td>
<td>40 100.25</td>
<td>43 163</td>
<td>0.3723</td>
<td>0.01039</td>
</tr>
<tr>
<td></td>
<td>06-2009</td>
<td>48 063.15</td>
<td>41 884</td>
<td>0.3140</td>
<td>0.00883</td>
</tr>
<tr>
<td>Philips</td>
<td>03-2009</td>
<td>10 769.76</td>
<td>4 534</td>
<td>0.5201</td>
<td>0.01482</td>
</tr>
<tr>
<td></td>
<td>06-2009</td>
<td>12 757.50</td>
<td>4 429</td>
<td>0.3691</td>
<td>0.01615</td>
</tr>
<tr>
<td>Siemens</td>
<td>03-2009</td>
<td>39 530.16</td>
<td>25 724</td>
<td>0.5321</td>
<td>0.01703</td>
</tr>
<tr>
<td></td>
<td>06-2009</td>
<td>45 015.38</td>
<td>23 753</td>
<td>0.3492</td>
<td>0.01197</td>
</tr>
<tr>
<td>Volkswagen</td>
<td>03-2009</td>
<td>72305.57</td>
<td>73 658</td>
<td>0.8688</td>
<td>0.039</td>
</tr>
<tr>
<td></td>
<td>06-2009</td>
<td>76 364.22</td>
<td>77 123</td>
<td>0.9608</td>
<td>0.0265</td>
</tr>
</tbody>
</table>

Table 1: equity market capitalization, debt (both figures in millions of euro) and implied volatility. Source: Bloomberg

The debt value reported in Table 1 is the sum of short and long-term debt. We depart from the standard KMV’s approach in which the value of the debt is the sum of the
short term debt and a half of the long term one. For each company we solve, using Excel’s Solver, the non-linear system of equations (4) using the March and June 2009 end-of-quarter data (Table 1). As a proxy for the risk-less rate, the one-year Euribor rate is used: at the end of March 2009 the rate was 0.01801 while at the end of June 2009 the rate was 0.01497.

Default probabilities have been recovered from the companies five-year CDS spreads, following the methodology presented in O’Kane and Turnbull (2003). In particular, we equate the present value of the periodical payments made by the protection buyer to the protection seller to the expected flow the protection seller is supposed to pay to the protection buyer in case default occurs. This equation is solved numerically with respect to $PD_{obs}$, the unknown market implied probability of the reference entity to become insolvent within a given period (one year in our analysis). The recovery rate, i.e. the portion of outstanding debt the creditor will recover in case of bankruptcy, has been set equal to 0.25.

3.2) Results and numerical issues

The numerical algorithm used to solve system (4) converges, finding a solution, for each company in both Industry and Utilities sectors (Table 2). Values of the curvature parameter obtained for these companies are always positive, ranging between 0.06 and 0.37. According to the Black and Cox model, the barrier is decreasing for increasing time to maturity. We claim that this result is an evidence that a time-dependant default barrier suits market data in a structural default model.

For one of the two companies in Automobile sector (Volkswagen), the algorithm does not converge. A possible explanation to this fact is that its implied volatility is
abnormally large. This is due to the exceptional turbulence financial market confronted during the entire 2009. It is then reasonable to claim that the second equation in system (4) forces the numerical routine not to converge. This idea seems to be confirmed for the Daimler case, the other company in the Automobile sector. In March 2009 the implied volatility of the equity is quite large and the routine does not converge. Three months later (June 2009) the implied volatility reduced significantly, becoming was small enough for the numerical routine to converge appropriately.

<table>
<thead>
<tr>
<th>Company</th>
<th>Date</th>
<th>Probability of default.</th>
<th>Barrier curvature</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bayer</td>
<td>March 2009</td>
<td>0.01143</td>
<td>0.23</td>
</tr>
<tr>
<td></td>
<td>June 2009</td>
<td>0.0078</td>
<td>0.28</td>
</tr>
<tr>
<td>Daimler</td>
<td>March 2009</td>
<td>0.0338</td>
<td>N.A.</td>
</tr>
<tr>
<td></td>
<td>June 2009</td>
<td>0.02264</td>
<td>0.06</td>
</tr>
<tr>
<td>Enel</td>
<td>March 2009</td>
<td>0.04382</td>
<td>0.45</td>
</tr>
<tr>
<td></td>
<td>June 2009</td>
<td>0.01615</td>
<td>0.31</td>
</tr>
<tr>
<td>E.on</td>
<td>March 2009</td>
<td>0.01039</td>
<td>0.17</td>
</tr>
<tr>
<td></td>
<td>June 2009</td>
<td>0.00883</td>
<td>0.31</td>
</tr>
<tr>
<td>Philips</td>
<td>March 2009</td>
<td>0.01482</td>
<td>0.36</td>
</tr>
<tr>
<td></td>
<td>June 2009</td>
<td>0.01615</td>
<td>0.37</td>
</tr>
<tr>
<td>Siemens</td>
<td>March 2009</td>
<td>0.01703</td>
<td>0.53</td>
</tr>
<tr>
<td></td>
<td>June 2009</td>
<td>0.01197</td>
<td>0.35</td>
</tr>
<tr>
<td>Volkswagen</td>
<td>March 2009</td>
<td>0.039</td>
<td>N.A.</td>
</tr>
<tr>
<td></td>
<td>June 2009</td>
<td>0.0265</td>
<td>N.A.</td>
</tr>
</tbody>
</table>

Table 2: implied one-year default probability and barrier curvature parameter

As a benchmark, it might be worth pointing out that implied volatility values above 0.6 must be considered outliers, if we believe that the 0.6 volatility level reached by the Chicago Board Options Exchange Volatility Index (VIX) calculated for S&P 500 index in the last months of 2008 is generally viewed as a remarkable peak.
Another convergence problem raises when trying to insert into system (4) debt as suggested by the KMV’s model. Replacing total debt with this reduced value ends up in no numerical solution. It results that the market probability of default is too large to be accommodated by the model. Our claim is that, under this point of view, KMV’s approach set a debt level that is too low to be considered a barrier, either constant or not constant.

4) Conclusions

In this article we want to empirically verify if the introduction of a time-dependant barrier into a structural model in which default is triggered when a stochastic process hits a barrier is plausible. To do this we apply a slightly simplified version of the Black and Cox structural model to a panel of companies listed in the Euro Stoxx 50 index. Our results find that barriers show a curvature. In some cases it is not possible to numerically solve the problem. We guess that this is due to the fact that some implied volatilities are very large; this could drive instability into the routine with, as a consequence, a lack of numerical convergence.

Bibliography


